MATH 407 Homework 12

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Ex.6.18 Since X_1, X_2 are independent, their joint probability is the product of marginals:

(1) $X_1X_2 = 0$ if and only if $X_1 = 0$ or $X_2 = 0$. This gives

$$\begin{aligned}P(X_1X_2 = 0) &= P(X_1 = 0) + P(X_2 = 0) - P(X_1 = X_2 = 0)\\ &= (1-p_1)^{n_1} + (1-p_2)^{n_2} - P(X_1 = 0)P(X_2 = 0)\\ &= (1-p_1)^{n_1} + (1-p_2)^{n_2} - (1-p_1)^{n_1}(1-p_2)^{n_2}.\end{aligned}$$

(2) If $X_1 + X_2 = 1$ then either $(X_1, X_2) = (0, 1)$ or the other way around:

$$\begin{aligned}P(X_1 + X_2 = 1) &= P(X_1 = 0, X_2 = 1) + P(X_1 = 1, X_2 = 0)\\ &= P(X_1 = 0)P(X_2 = 1) + P(X_1 = 1)P(X_2 = 0)\\ &= (1-p_1)^{n_1} \cdot n_2p_2(1-p_2)^{n_2-1} + n_1p_1(1-p_1)^{n_1-1}(1-p_2)^{n_2}\\ &= (1-p_1)^{n_1-1}(1-p_2)^{n_1-1}\left[(1-p_1)n_2p_2 + (1-p_2)n_1p_1\right].\end{aligned}$$

(3) We can have $(X_1X_2) = (0, 2), (1, 1)$, or $(2, 0)$. Thus

$$\begin{aligned}P(X_1 + X_2 = 2) &= P(X_1 = 0)P(X_2 = 2) + P(X_1 = 1)P(X_2 = 1) + P(X_1 = 2)P(X_2 = 0)\\ &= (1-p_1)^{n_1} \cdot \frac{n_2(n_2-1)}{2}p_2^2(1-p_2)^{n_2-2} + n_1p_1(1-p_1)^{n_1-1}n_2p_2(1-p_2)^{n_2-1}\\ &\quad + \frac{n_1(n_1-1)}{2}p_1^2(1-p_1)^{n_1-2} \cdot (1-p_2)^{n_2}\\ &= (1-p_1)^{n_1-2}(1-p_2)^{n_2-2}k\end{aligned}$$

where

$$k = \frac{n_2(n_2-1)}{2}(1-p_1)^2p_2^2 + n_1p_1(1-p_1)n_2p_2(1-p_2) + \frac{n_1(n_1-1)}{2}p_1^2(1-p_2)^2.$$

Ex.6.27

$$\begin{aligned}P(X_1/X_2 < a) = P(X_1 < X_2a) &= \int_0^\infty \int_0^{ay} f_{X_1,X_2}(x, y)\,\mathrm{d}x\,\mathrm{d}y\\ \text{[independence]} &= \int_0^\infty \int_0^{ay} \lambda_1\lambda_2e^{-\lambda_1x}e^{-\lambda_2y}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^\infty \lambda_1\lambda_2e^{-\lambda_2y} \int_0^{ay} e^{-\lambda_1x}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^\infty \lambda_1\lambda_2e^{\lambda_2y}(1 - e^{-\lambda_1ay})\,\mathrm{d}y\\ &= \lambda_2 \int_0^\infty \lambda_1e^{-\lambda_2y} - e^{-y(\lambda_1+\lambda_2a)}\,\mathrm{d}y\\ &= \frac{\lambda_1a}{\lambda_2 + \lambda_1a}.\end{aligned}$$

Therefore

$$f_Z(a) = \frac{\mathrm{d}}{\mathrm{d}a}\frac{\lambda_1 a}{\lambda_2 + \lambda_1 a} = \frac{\lambda_1\lambda_2}{(\lambda_1 a + \lambda_2)^2}.$$

It follows immediately that

$$P(X_1/X_2 < 1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Ex.6.32 (1) The probability to have a sale > 100 in a month is precisely $1/2$. Thus the total probability is

$$\binom{6}{3} \cdot (1/2)^3 \cdot *(1/2)^{6-3} = \frac{5}{16}.$$

(2) If $X_1, \ldots X_4 \sim N(100, 25)$ and are i.i.d., then $Y := \sum X \sim N(400, 100)$. Therefore the standard distribution of such a sum variable is 10 and 420 is 2 standard deviations away. The probability is therefore

$$P(Y > 440) = P(Z > 2) = 0.02275.$$

Th.6.5 This is somewhat similar to the second problem in this problem set.

$$\begin{aligned} P(X/Y < a) = P(X < aY) &= \int_{-\infty}^{\infty} P(X < ay)\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{ay} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{ay} f_X(ay) f_Y(y)\,\mathrm{d}x\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} F_X(x) f_Y(y)\,\mathrm{d}y \end{aligned}$$

and (assuming $y \neq 0$)

$$P(XY < a) = P(X < a/Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{a/y} f_X(ay) f_Y(y)\,\mathrm{d}x\,\mathrm{d}y = \int_{-\infty}^{\infty} f_X(a/y) f_Y(y)\,\mathrm{d}y.$$

It follows that they respectively have PDF

$$\int_{-\infty}^{\infty} f_X(ay) y f_Y(y)\,\mathrm{d}y \text{ and } \int_{-\infty}^{\infty} \frac{f_X(a/y) f_Y(y)}{y}\,\mathrm{d}y.$$

If X, Y are exponential random variables then the first one (by the previous problem) has PDF

$$f_{X/Y}(a) = \frac{\lambda_1\lambda_2}{(\lambda_1 a + \lambda_2)^2}$$

and the second one is

$$\int_0^{\infty} \frac{\lambda_1 e^{-\lambda_1 a/y}\lambda_2 e^{-\lambda_2 y}}{y}\,\mathrm{d}y.$$

Th.6.7 If $X \sim \text{Gamma}(t, \lambda)$ then

$$F_X(x) = \frac{\lambda}{\Gamma(t)}\int_0^{\infty} e^{-\lambda x}(\lambda x)^{t-1}\,\mathrm{d}x.$$

Thus,

$$F_{cX}(x) = F_X(x/c) = \frac{\lambda}{\Gamma(t)}\int_0^{\infty} e^{-\lambda x/c}(\lambda x/c)^{t-1}\,\mathrm{d}x$$

and differentiating this gives

$$f_{cX}(x) = \frac{\lambda}{\Gamma(t)} e^{-(\lambda/c)x}((\lambda/c)x)^{t-1}.$$

Therefore cX is simply a Gamma random variable with parameters $(t, \lambda/c)$.