MATH 407 Homework 13

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Ex.7.36 It is hard to deal with X and Y directly, but we can write $X := \sum_{i=1}^n X_i$ and $Y := \sum_{i=1}^n Y_i$ where X_i, Y_i evaluates to 1 if the i^{th} roll is 1 or 2 respectively and 0 otherwise. Immediately we see that when $i \neq j$, X_i and Y_j are independent. Therefore, by the bilinearity of $\text{Cov}(\cdot,\cdot)$,

$$\text{Cov}(X,Y) = \text{Cov}\left(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i\right) = \sum_{i,j=1}^n \text{Cov}(X_i, Y_i) = \sum_{i=1}^n \text{Cov}(X_i, Y_i).$$

For each term like this,

$$\text{Cov}(X_i, Y_i) = E[X_i Y_i] - E[X_i]E[Y_i]$$

where $E[X_i Y_i]$ is clearly 0 since $X_i Y_i \neq 0$ only when $X_i = 1$ and $Y_i = 2$, but then such probability is 0. It is also clear that $E[X_i] = E[Y_i] = 1/6$. Therefore

$$\text{Cov}(X,Y) = \sum_{i=1}^n \text{Cov}(X_i, Y_i) = \frac{n}{36}.$$

Ex.7.42 The y-marginal is fairly easy to compute:

$$f_Y(y) = \int_0^\infty \frac{e^{-(y+x/y)}}{y}\,\mathrm{d}y = \int_{-\infty}^{-y} e^{-u}\,\mathrm{d}u = e^{-y}.$$

Clearly $E[Y] = \int_0^\infty ye^{-y}\,\mathrm{d}y = 1$. On the other hand,

$$E[X] = E[E[X \mid Y]] = \int_0^\infty x \cdot \frac{f_{X,Y}(x,y)}{f_Y(y)}\,\mathrm{d}x = \int_0^\infty \frac{x\exp(-y-x/y)/y}{e^{-y}}\,\mathrm{d}x = 1.$$

Finally,

$$\begin{aligned}
\text{Cov}(X,Y) &= E[XY] - E[X]E[Y] \\
&= E[E[XY \mid Y]] - E[X]E[Y] \\
&= E[YE[X \mid Y]] - E[X]E[Y] \\
&= E[Y^2] - E[X]E[Y] \\
&= 2 - 1 = 1.
\end{aligned}$$

TE.7.27 Once X is given, $g[X]$ is simply a constant.

TE.7.30 Since

$$E[X_1 + ... + X_n \mid X_1 + ... + X_n = x] = x,$$

and each X_i is i.i.d., we have

$$E[X_1 \mid X_1 + ... + X_n = x] = \frac{x}{n}.$$

TE.7.33

$$\begin{aligned} E[X \mid A] &= \sum_{x \in X} P(x \mid A) = \sum_{x \in X} \frac{P(\{x\} \cap A)}{P(A)} \\ &= \frac{1}{P(A)} \left[\sum_{x \in A} 1 \cdot P(x) + \sum_{x \notin A} 0 \cdot P(x) \right] = \frac{E[XI_A]}{P(A)}. \end{aligned}$$