MATH 407 Homework 2

Qilin Ye

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Chapter 2, Problems

Ex.2.1 $\Omega = \{\text{RR, RG, RB, GR, GG, GB, BR, BG, BB}\}$ and $\Omega' = \{\text{RG, RB, GR, GB, BR, BG}\}$, where R,G,B stand for red, green, and blue, respectively.

Ex.2.2 The sample space is

$$\Omega = \bigcup_{k\geqslant 1}\{x_1,\dots,x_{k-1},6 \mid 1\leqslant x_i\leqslant 5\}$$

and

$$E_n = \bigcup_{k\geqslant n+1}\{x_1,\dots,x_{k-1},6 \mid 1\leqslant x_i\leqslant 5\}.$$

The complement of $\bigcup E_n$ is the set of events in which we never get 6, i.e., the set of infinite sequences of form $\{x_n\}\subset\{1,2,3,4,5\}$.

Ex.2.3 For convenience, we first write $F = \{(1,1),(1,2),\dots,(1,6),(2,1),\dots,(6,1)\}$ and $G = \{(1,4),(2,3),(3,2),(4,1)\}$.

$$\begin{aligned}
E\cap F &= \{(1,2),(1,4),(1,6),(2,1),(4,1),(6,1)\}\\
E\cup F &= \{(1,1),(1,2),\dots,(1,6),(2,1),\dots,(6,1),\quad (2,3),(2,5),\\
&\qquad (3,2),(3,4),(3,6),(4,3),(4,5),(5,2),(5,4),(5,6),(6,3),(6,5)\}\\
F\cap G &= \{(1,4),(4,1)\}\\
E\cap F^c &= \{(2,3),(2,5),(3,2),(3,4),(3,6),(4,3),(4,5),(5,2),(5,4),(5,6),(6,3),(6,5)\}\\
E\cap F\cap G &= \{(1,4),(4,1)\}
\end{aligned}$$

Ex.2.5 For (a), $|\Omega| = 2^5 = 32$. For (b),

$$W = W_1\cup W_2\cup W_3 \text{ where } \begin{cases} W_1 = \{1,1,x_3,x_4,x_5 \mid x_i\in\{0,1\}\}\\ W_2 = \{x_1,x_2,1,1,x_5 \mid x_i\in\{0,1\}\}\\ W_3 = \{1,x_2,1,x_4,1 \mid x_i\in\{0,1\}\}\end{cases}$$

For (c), there are no constraints on the first three events $\implies |A| = 2^3 = 8$.

For (d), $A\cap W = \{(1,1,1,0,0),(1,1,0,0,0)\}$.

Ex.2.6 For (a), $\Omega = \{(x,a) \mid x \in \{0,1\}, a \in \{\text{`g'}, \text{`f'}, \text{`s'}\}\}$.

For (b), $A = \{(1,s),(0,s)\}$.

For (c), $B = \{(0,g),(0,f),(0,s)\}$.

For (d), $B^c \cup A = \{(1,g),(1,f),(1,s),(0,s)\}$.

Ex.2.8 (a) 0.8; (b) 0.3; (c) 0.

Ex.2.9 $24\% + 61\% - 11\% = 74\%$.

Ex.2.12 For (a), we compute the total number of students using inclusion-exclusion:

$$28 + 26 + 16 - 12 - 4 - 6 + 2 = 50 \implies \text{also 50 not taking} \implies P = 0.5.$$

For (b), we adjust the inclusion-exclusion slightly:

$$28 + 26 + 16 - 2(12 + 4 + 6) + 3 \cdot 2 = 32 \implies P = 0.32.$$

For (c), calculating the complement gives 0.5^2 and so the actual answer is $1 - 0.5^2 = 0.75$.

Ex.2.25 Let A_5 denote the events in which 5 occurs first and A_7 the events in which 7 occurs first. Since

$$P(A_5) = \frac{|A_5|}{|\Omega|} \text{ and } P(A_7) = \frac{|A_7|}{|\Omega|},$$

the probability that A_5 occurs first is the same as asking $P(A_5)/(P(A_5 + P(A_7))$, because it is equivalent to asking *if one between A_5 and A_7 happens, what's the probability that A_5 is the one that happened?* Then this problem reduces to $|A_5|/(|A_5| + |A_7|) = 4/(4+6) = 0.4$.

Chapter 2, Theoretical Exercises

Ex.2.1* $E \cap F = \{x \mid x \in E \wedge x \in F\} \implies E \cap F \subset E$. Similarly, $E \cup F = \{x \mid x \in E \vee x \in F\} \implies E \subset E \cup F$.

Ex.2.2* $F^c \subset E^c$ is simply the contrapositive of $E \subset F$.

Ex.2.3*

$$\begin{aligned} x \in (F \cap E) \cup (F \cap E^c) &\iff x \in F \cap E \text{ or } x \in F \cap E^c \\ &\iff (x \in F \text{ and } x \in E) \text{ or } (x \in F \text{ and } x \notin E) \\ &\iff x \in F \text{ and } (\cancel{x \in E \text{ or } x \notin E}) \end{aligned}$$

and

$$\begin{aligned} x \in E \cup F &\iff x \in E \text{ or } x \in F \\ &\iff x \in E \text{ or } (x \notin E \text{ but } x \in F) \\ &\iff x \in E \text{ or } x \in E^c \cap F \\ &\iff x \in E \cup (E^c \cap F). \end{aligned}$$

Ex.2.5* Define inductively with $F_1 := E_1$ and $F_n := \bigcup_{i=1}^{n} E_i \setminus \bigcup_{i=1}^{n-1} E_i$.

Ex.2.18* Indeed, for f_2 as long as we don't have heads twice we are fine, and obviously $f_2 = 3$. For $n \geqslant 3$, it remains to notice that every sequence of n tosses without consecutive heads is uniquely determined by appending a T to a sequence of $n-1$ tosses or appending TH to a sequence of $n-2$ sequences. Hence $f_n = f_{n-1} + f_{n-2}$. Notice that $f_{10} = 144$ and so $P_{10} = 144/1024 = 9/64$.

Ex.2.20* Suppose not. If $P(\{x\}) = 0$ for all $x \in \Omega$ then $P(\Omega) = \sum_{x\in\Omega} P(\{x\}) = 0$, contradicting one of the axioms. Since $P(\{x\}) \geqslant 0$ by another axiom, the only remaining case is if $P(\{x\}) = c > 0$, but then $P(\Omega) = \sum_{x\in\Omega} P(\{\Omega\}) = \infty$, contradicting some axiom once again. Hence they can't be all equally likely.

As for the second question — yes. Consider the experiment that maps a random number in $(0,1]$ to $-\lfloor \log_2(x) \rfloor$, a surjection onto $\mathbb{N}$, with $P(\{n\}) = 2^{-n}$ for all $n \in \mathbb{N}$.