MATH 407 Problem Set 3

Qilin Ye

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Remark. For simplicity of notation, I will denote *permutation* and *combination* of n choose r by

$$P_n^r := \frac{n!}{(n-r)!} \text{ and } C_n^r := \frac{n!}{r!(n-r)!}, \text{ respectively.}$$

Chapter 1 Problems

Prob.1.8 If a word of length n has no repeating letters then the answer is simply $n!$. If, say, a letter appears twice in the word, then one should divide the factorial by 2! because order of identical letters doesn't matter. Now back to the question: *Fluke* simply has $5! = 120$ arrangements. *Purpose* has a pair of p and a pair of o so the answer is $7!/(2!2!) = 1260$. *Mississippi* has four i's, four s's, and a pair of p. It's total length is 11. The answer is therefore $11!/(4!4!2!) = 34650$. *Arrange* is the same as *Purpose*.

Prob.1.10

(a) Simply $8! = 40320$.

(b) There are 14 ways to arrange these two people. The other 6 can sit freely once A and B have sat down. The answer is $14 \cdot 6! = 10080$.

(c) If men sit at odd numbers then we have $4! \cdot 4!$ arrangements. If women sit at odd numbers we have another $4! \cdot 4!$. Thus the total number is $2 \cdot 4! \cdot 4! = 1152$.

(d) $5! = 120$ ways to arrange the 5 men. Then the rest can sit freely. $5! \cdot 3! = 720$.

(e) Treating each couple as a "block" first, we have 4 blocks and 24 arrangements for these blocks. Within each block there are 2 possibilities. $24 \cdot 2^4 = 384$.

Prob.1.11

(a) $6! = 720$.

(b) Use the "block" strategy again: $3! \cdot (1!2!3!) = 72$.

(c) Still the "block" strategy but with 4 blocks (one for novels, one for each remaining book): $4!3! = 144$.

Prob.1.15 This is equivalent to saying "how many pairs between 20 people can be made", of which the answer is $C_{20}^2 = 190$.

Prob.1.23 This is equivalent to asking how many arrangements of 4 R's and 3 U's are there, of which the answer is $C_7^3 = 35$.

Prob.1.26

(a) Consider a set of n elements, all of which are to be numbered $1, 2$ or 3. There should be a total of 3^n ways to do so. On the other hand, we can start case by case, based on how many elements are to be numbered 3. Say $n - k$. Then there are $C_n^{n-k} = C_n^k$ ways to pick these elements, and for the rest k elements, each is to be numbered either 1 or 2, resulting in 2^k ways. Hence in each case we have $C_n^k \cdot 2^k$ ways, and letting k vary between 0 and n, inclusive, gives the equality.

(b) $(x+1)^n$, similar reasoning as above: simply replace $\{1, 2, 3\}$ by $\{1, 2, \cdots, x+1\}$.

Prob.1.33 We first treat England and France as a "block" and ignore Russia and U.S. Then we have 7 entities that are to be randomly arranged, resulting in 7! options. Now within the Anglo-French block there are $2! = 2$ ways to arrange. Finally, let the Russian and American come. These 7 entities create 8 spaces for these two to choose, and since order matters, there are $P_8^2 = 56$ options. Multiplying everything together we get $7! \cdot 2 \cdot 56 = 564480$ ways.

Chapter 1 Theoretical Exercises

Ex.1.8 (*Vandermonde's Identity.*) **Assuming $\boldsymbol{r \leqslant \min\{m, n\}}$**, choosing r elements from a total of $m + n$ is equivalent to choosing k elements from m and $r - k$ from n where $0 \leqslant k \leqslant r$. Hence

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}.$$

Ex.1.9 This problem follows from (8) by realizing that $\binom{n}{k} = \binom{n}{n-k}$. Then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Ex.1.11 The number of subsets of k elements with i as the largest index is $\binom{i-1}{k-1}$ (since one element is determined). The index $i - 1$ can range from $k - 1$ to $n - 1$ and thus

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}.$$

Ex.1.12

(a) There are n ways to pick a chairperson and, for the remaining $n - 1$ people, each can be either chosen or not, hence the quantity $n \cdot 2^{n-1}$. On the other hand, we can also get the same result by first fixing a $k \in [1, n]$. Choosing k people to form a group out of n people gives C_n^k options, whereas choosing one chairperson among this group gives k options. Multiplying them together, and letting k index through 1 to n gives the other side of the equation.

(b) On one hand, for a group of k people, there will be $\binom{n}{k}k^2$ ways, where the k^2 comes from picking two people from the group without replacement. On the other hand, if the chairperson and the secretary are the same, there are $n2^{n-1}$ ways to form a group, where n comes from randomly selecting a chairperson / secretary, and 2^{n-1} comes from putting or not putting any remaining person into the group. If the chairperson and the secretary are by different people, then there are $n(n-1)$ ways to choose these two, after which there are 2^{n-2} ways to decide if the remaining $n-2$ will join the group. Hence the number $n(n-1)2^{n-2}$. Adding them together, we have

$$n2^{n-2} + n(n-1)2^{n-2} = n(n+1)2^{n-2}.$$

(c) Now we want to choose a committee of arbitrary size out of n people and choose three special roles (chair, secretary, and vice-chair, say), and we allow the same person to have more than one role. On onehand, for such a committee of k people we have $k^3C_n^k$ options, so letting k vary gives a total of

$$\sum_{k=1}^{n}\binom{n}{k}k^3.$$

On the other hand, if all three roles are assigned to the same person, there are $n2^{n-1}$ possibilities. If one person holds two and someone else holds the third, there are $3n(n-1)2^{n-2}$ possibilities. If three roles are held by 3 different people, there are $n(n-1)(n-2)2^{n-3}$ ways. Adding them together we have

$$n2^{n-1} + 3n(n-1)2^{n-2} + n(n-1)(n-2)2^{n-3} = 2^{n-3}n^2(n+3)$$

possibilities. Since the two sides must equal, we have derived the equality we are asked to show.

Ex.1.18 Indeed, let the n^{th} element be picked first. It has to belong to one of the members of the partitions, say n_i. Then with the assumption that this n^{th} element belongs to n_i we have

$$\binom{n}{n_1,\ldots,n_r} = \binom{n-1}{n_1,\ldots,n_i-1,\ldots,n_r}.$$

Since this n^{th} element can also be in other partition members, we let i range through 1 to r and obtain the sum:

$$\binom{n}{n_1,\ldots,n_r} = \sum_{i=1}^{r}\binom{n-1}{n_1,\ldots,n_i-1,\ldots,n_r}.$$

Chapter 1, Self Test Problems

Prob.1.11

(a) $C_{10}^6 \cdot 2^6 = 13440$ where the binomial coefficient comes from picking 6 out of 10 and 2^6 comes from picking either the man or the woman from each couple.

(b) The first term is still C_{10}^6. Once we've decided which couples to pick from, we have $C_6^3 = 20$ ways to decide from which three couples do these three men come from (so the remaining three must be women). The answer is therefore $C_{10}^6 \cdot C_6^3 = 4200$.

Prob.1.12 A committee of 6 people with at least 3 women and 2 men is a very strong restriction. We have thus only two possibilities to consider.

(1) 4 women and 2 men: $C_8^4 \cdot C_7^2$ ways.

(2) 3 women and 3 men: $C_8^3 \cdot C_7^3$ ways.

They together provide 3430 ways.

Prob.1.17

(a) Analytic:

$$\begin{aligned}\binom{k}{2} + k(n-k) + \binom{n-k}{2} &= \frac{k(k-1)}{2} + k(n-k) + \frac{(n-k)(n-k-1)}{2} \\ &= \frac{k^2 - k + 2kn - 2k^2 + n^2 - 2nk + k^2 - n + k}{2} \\ &= \frac{n^2 - n}{2} = \frac{n(n-1)}{2} = \binom{n}{2}.\end{aligned}$$

(b) Combinatorial: to pick 2 elements from a group of n elements, one can either pick both from the first k elements, giving C_k^2 options, or pick both from the last $n-k$ elements, giving C_{n-k}^2 options, or pick one from the first k and one from the remaining $n-k$, giving $k(n-k)$ options.

Prob.1.21 Notice that

$$0 = ((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = (-1) \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1}.$$

Adding 1 to the LHS and removing the term $k = 0$ gives our desired equality

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} = 1.$$

Chapter 2, Problems

Ch.2.15

(a) 4 ways to choosen the suit and C_{13}^5 ways to choose the denominations. Hence there are $4 \cdot C_{13}^5$ ways out of C_{52}^5 ways, i.e., $P = 4 \cdot C_{13}^5 / C_{52}^5$.

(b) 13 ways to choose the denominator of the pair. C_4^2 combinations of suits within the pair. C_{12}^3 ways to choose b, c, d, and 4^3 ways to choose their suits. $P = 13 \cdot 4^3 \cdot C_4^2 C_{12}^3 / C_{52}^5$.

(c) C_{13}^2 ways to choose the denominator of the pairs; C_4^2 ways of choosing suits for each pair. 11 ways to choose the remaining c and 4 ways for its suit. Hence $P = 44 \cdot C_{11}^2 (C_4^2)^2 / C_{52}^5$.

(d) 13 ways to choose the denominator of the triplet; C_4^3 ways for the suits. C_{12}^2 ways to choose the remaining denominators with 4^2 suits. $P = 4^2 \cdot 13 \cdot C_{12}^2 C_4^3 / C_{52}^5$.

(e) 13 ways to choose the denominator of the quadruplet and the $C_4^4 = 1$ way to choose the suit. $12 \cdot 4 = 48$ ways to choose the remaining card. $P = 13 \cdot 48 / C_{52}^5$.

Prob.2.40

(a) Consider the complement: no ball is green, which there are C_{15}^4 possibilities. Hence $P = 1 - C_{15}^4 / C_{22}^4$.

(b) Note that we only pick 4 balls and one of each four colors must be chosen. Hence we directly have the total number of possibilities: $4 \cdot 5 \cdot 6 \cdot 7$. Divide this by C_{22}^4 to get the probability.