MATH 407 HW4

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Chapter 3 Exercises

3.17 $P(\text{survive} \mid \text{C}) \cdot P(C) + P(\text{survive} \mid \text{no C}) P(\text{no C}) = 0.96 \cdot 0.15 + P(\text{survive} \mid \text{no } C) \cdot 0.85 = 0.8$
$\implies P(\text{survive} \mid \text{no } C) \approx 0.983.$

3.19

$$P(\text{ind} \mid \text{voted}) = \frac{P(\text{voted} \mid \text{ind})}{P(\text{voted})} = \frac{0.35 \cdot 0.46}{0.35 \cdot 0.46 + 0.62 \cdot 0.3 + 0.58 \cdot 0.24} \approx 0.331.$$

$$P(\text{lib} \mid \text{voted}) = \frac{P(\text{lib} \mid \text{voted})}{P(\text{voted})} = \frac{0.62 \cdot 0.3}{\sum} \approx 0.383.$$

$$P(\text{con} \mid \text{voted}) = \frac{P(\text{con} \mid \text{voted})}{P(\text{voted})} = \frac{0.58 \cdot 0.24}{\sum} \approx 0.286.$$

The total percentage is simply the denominator above, 0.486.

3.45

$$P(\text{two} \mid H) = \frac{P(H \mid \text{two}) P(\text{two})}{P(H)} = \frac{1 \cdot 1/3}{0.5 \cdot 1/3 + 1/3 + 3/4 \cdot 1/3} = \frac{4}{9}.$$

3.57 (a)

$$P(A \text{ leads} \mid 3:0) = \frac{P(A \text{ leads with } 3:0)}{P(A3:B0) + P(A0:B3)} = \frac{p^3}{p^3 + (1-p)^3}.$$

(b) Let $P(X|Y)$ be the probability that the leading team wins, given Y, the event that leading team leads 3 to 0. Let X_A, X_B be the events where A or B wins, respectively, and let Y_A, Y_B be the events where A or B leads 3 to 0, respectively. Notice that $X_A \cap X_B = Y_A \cap Y_B = \varnothing$ but $X = X_A \cup X_B$ and $Y = Y_A \cup Y_B$. Then,

$$\begin{aligned}
P(X|Y) &= \frac{P(X \cap Y)}{P(Y)} \\
&= \frac{P(X_A \cap Y) + P(X_B \cap Y)}{P(Y)} \\
&= \frac{P(X_A \cap Y_A) + P(X_B \cap Y_B)}{P(Y)} \\
&= \frac{P(X_A|Y_A)P(Y_A) + P(X_B|Y_B)P(Y_B)}{P(Y_A) + P(Y_B)} \\
&= \frac{(1-(1-p)^4)p^3 + (1-p^4)(1-p)^3}{p^3 + (1-p)^3}.
\end{aligned}$$

3.60 Suppose there are n sophomore girls. As a shorthand notation, denote "boy", "girl", "freshman", and "sophomore" by their first letters respectively. Then the independence implies $P(G|F) = P(G)$. Notice that if this equation holds then replacing B or F (or both) by G and S won't change the result. For example,

$$P(B|F) = 1 - P(G|F) = 1 - P(G) = P(B) \text{ and } P(F|B) = \frac{P(FB)}{P(B)} = \frac{P(B)P(F)}{P(F)} = P(F).$$

To solve $P(G|F) = P(G)$:

$$P(G|F) = \frac{P(F|G)P(G)}{P(F)} = \frac{6/(6+n) \cdot (6+n)/(16+n)}{10/(16+n)} = 0.6$$

whereas $P(G) = (6+n)/(16+n)$. Therefore $6 + n = 0.6(16+n)$ and the solution is $n = 9$.

Chapter 3 Theoretical Exercises

3.5 (a) follows from Bayes' rule:

$$\begin{aligned} P(E|E \cup F) &= \frac{P(E \cup F|E)P(E)}{P(E \cup F)} \\ &= \frac{P(E)}{P(E \cup F)} && \text{since } P(E \cup F|E) = 1 \text{ is trivially true} \\ &= \frac{P(E)}{P(E) + P(F)} && \text{since } E \cap F = \varnothing. \end{aligned}$$

(b) holds similarly:

$$\begin{aligned} P(E_j | \textstyle\bigcup_{i=1}^\infty E_i) &= \frac{\overbrace{P(\bigcup_{i=1}^\infty E_i | E_j)}^{=1} P(E_j)}{P(\bigcup_{i=1}^\infty E_i)} \\ &= \frac{P(E_j)}{P(\bigcup_{i=1}^\infty E_i)} \\ &= \frac{P(E_j)}{\sum_{i=1}^\infty P(E_i)}. && \text{pairwise disjoint} \implies P(\textstyle\bigcup E_i) = \sum P(E_i) \end{aligned}$$

3.22 Let $\varphi(n)$ be the statement that $P_n = \frac{1}{2} + \frac{1}{2}(2p-1)^n$. Clearly $\varphi(1)$ holds since $P_1 = (2p-1)P_0 + (1-p) = p$ and

$$\frac{1}{2} + \frac{1}{2}(2p-1)^1 = \frac{1}{2} + p - \frac{1}{2} = p.$$

For the inductive step, assume $\varphi(k)$ holds. It follows that

$$\begin{aligned} P_{k+1} &= (2p-1)P_k + (1-p) \\ &= (2p-1)\left[\frac{1}{2} + \frac{1}{2}(2p-1)^k\right] + (1-p) \\ &= p - \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1} + 1 - p \\ &= \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1}, \end{aligned}$$

and so $\varphi(k) \implies \varphi(k+1)$, and the claim follows from this induction.