MATH 407 HW5

Qilin Ye

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Ch.3 Theoretical Exercises

3.16 To have an even number of success after n Bernoulli trials, one either needs to have a failure in the n^{th} trial with an even amount of successes in the first $n-1$ trials or a successful n^{th} trial with an odd number of successes in the first $n-1$ trials. Translating this into math language we have

$$P_n = (1-p)P_{n-1} + p(1-P_{n-1}).$$

For the second part, clearly $P_1 = 1-p$ since the only possibility to have an even number of successes is if there's no success at all. Indeed

$$\frac{1+(1-2p)^1}{2} = 1-p.$$

Now suppose P_k is correctly given by $P_k = \dfrac{1+(1-2p)^n}{2}$. Then,

$$\begin{aligned}
P_{k+1} &= (1-p)P_k + p(1-P_k) \\
&= (1-p)\frac{1+(1-2p)^n}{2} + p\frac{1-(1-2p)^n}{2} \\
&= \frac{1}{2}\left[1-p+(1-p)(1-2p)^n + p - p(1-2p)^n\right] \\
&= \frac{1}{2}\left[1-p+p+(1-2p)(1-2p)^n\right] \\
&= \frac{1+(1-2p)^{n+1}}{2}
\end{aligned}$$

and the claim follows from induction.

3.18 In order for n tosses to have no 3 consecutive heads, the last tail must appear no earlier than the $(n-2)^{\text{th}}$ toss. Therefore Q_n is given by

$$\begin{aligned}
Q_n &= P(n^{\text{th}}\ \text{T})Q_{n-1} + P(n^{\text{th}}\ \text{H},\ (n-1)^{\text{th}}\ \text{T})Q_{n-2} + P(n^{\text{th}}, (n-1)^{\text{th}}\ \text{H}, (n-2)^{\text{th}}\ \text{T})Q_{n-3} \\
&= \frac{1}{2}Q_{n-1} + \frac{1}{4}Q_{n-2} + \frac{1}{8}Q_{n-3}.
\end{aligned}$$

Then brute force computation suggests $Q_3 = 7/8, Q_4 = 13/16, Q_5 = 3/4, Q_6 = 11/16, Q_7 = 81/128$, and $Q_8 = 149/256$.

Ch.3 Self-Test Problems

3.8 $\dfrac{P(H|E)}{P(G|E)} = \dfrac{P(H)P(E|H)/P(E)}{P(G)P(E|G)/P(E)} = \dfrac{P(H)}{P(G)}\dfrac{P(E|H)}{P(E|G)}$ by applying Bayes' rule twice. If the first quotient $\dfrac{P(H)}{P(G)} =$ 3 and the second $= 1/2$, then the total ratio becomes $3/2 > 1$, so still H is more likely to happen.

Ch.4 Problems

4.7 The maximum can be any integer ranging from 1 to 6, inclusive, and the same for minimum. The sum should be from 2 to 12, inclusive, and (d) can be anything integer between -5 and 5, inclusive again.

4.11 $\sum_{i=1}^{9} p_{\mathcal{X}}(i) = \sum_{i=1}^{9} \log_{10}(i+1)/i = \log_{10}\left[\prod_{i=1}^{9}(i+1)/i\right] = \log_{10}(10/1) = 1$, and

$$P(\mathcal{X} \leqslant j) = \sum_{i=1}^{j} \log_{10}\frac{i+1}{i} = \log_{10}\left[\prod_{i=1}^{j}(i+1)/i\right] = \log_{10}(j+1).$$

Ch.4 Theoretical Exercises

4.2 Simply take the inverse of exp, i.e., $P(e^X \leqslant x) = P(X \leqslant \log x) = F(\log x)$.

4.3 Again, take the inverse, i.e., $P(\alpha X + \beta \leqslant x) = P(X \leqslant (x-\beta)/\alpha) = F((x-\beta)/\alpha)$ if $\alpha > 0$ and $P(\alpha X + \beta \leqslant x) = P(X \geqslant (x-\beta)/\alpha) = 1 - F((x-\beta)/\alpha)$ if $\alpha < 0$.