# MATH 407 HW6

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## Ch.4 Problems

4.19 The probably mass function is given by $p(0) = 1/2, p(1) = 1/10, p(2) = 1/5, p(3) = 1/10, p(3.5) = 1/10$, and 0 for all other $x \in \mathbb{R}$.

## Ch.4 Theoretical Exercises

4.10

$$\begin{aligned} E[1/(X+1)] &= \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \frac{n!}{(n-k)!(k+1)!p} \cdot p^{k+1}(1-p)^{n-k} \\ &= \frac{1}{(n+1)p} \sum_{k=1}^{n+1} \frac{(n+1)!}{(n+1-k)!k!} \left[p^k (1-p)^{n+1-k}\right] \\ &= \frac{1}{(n+1)p} \cdot (1-(1-p)^{n+1}). \end{aligned}$$

4.28 Since $X$ is a geometric random variable, the sequence $\{p_n\}$ defined by $p_n := P(X = n)$ form a geometric sequence, and the relative ratio between $p_n$ and $p_m$ depends only on how far are they away from each other, i.e., $|m - n|$, not where $m$ and $n$ are exactly. Therefore the probability of $X = n + k$ given $X > n$ is the same as the probability of $X = (n + k) - n$ given $X > (n - n)$, as long as they both sense, which they indeed do (and the latter is simply "the probability of $X = k$", since "$X > n - n = 0$" is vacuously true).

$$\begin{aligned} P(X = n+k \mid X > n) &= P(X = n+k)/P(X > n) \\ &= p(1-p)^{n+k-1} \Big/ \sum_{i=n}^{\infty} p(1-p)^i \\ &= p(1-p)^{n+k-1} / [p(1-p)^n/p] \\ &= p(1-p)^{k-1} = P(X = k). \end{aligned}$$

4.30 Assuming the parameters are $n, N, m$:

$$\begin{aligned} P(X = k+1)/P(X = k) &= \left[\binom{m}{k+1}\binom{N-m}{n-k-1}\binom{N}{n}\right] \Big/ \left[\binom{m}{k}\binom{N-m}{n-k}\binom{N}{n}\right] \\ &= \left[\frac{m!}{(k+1)!(m-k-1)!}\frac{k!(m-k)!}{m!}\right]\left[\frac{(N-m)!}{(N-m-n+k+1)!(n-k-1)!}\frac{(N-m-n+k)!(n-k)!}{(N-m)!}\right] \\ &= \frac{m-k}{k+1} \cdot \frac{n-k}{N-m-(n-k-1)}. \end{aligned}$$

4.37 (a) To have an event where $X+Y$ evaluates to $z_k$ we need some $\tilde{x} \in \{x_i\}, \tilde{y} \in \{y_i\}$ such that $\tilde{x}+\tilde{y} = z_k$. Summing all possible scenarios together we have the desired equality.

(b)

$$\begin{aligned} E[X+Y] &= \sum_{\tilde{z}\in\{z_k\}} [\tilde{z}\cdot P(X+Y=\tilde{z})] \\ &= \sum_{\tilde{z}\in\{z_k\}} \left[\tilde{z}\cdot \sum_{(i,j)\in A_k} P(X=x_i, Y=y_j)\right] \\ &= \sum_{\tilde{z}\in\{z_k\}} \sum_{(i,j)\in A_k} [\tilde{z}\cdot P(X=x_i, Y=y_i)] \\ &= \sum_{\tilde{z}\in\{z_k\}} \sum_{(i,j)\in A_k} [(x_i+y_j)P(X=x_i, Y=y_j)]. \end{aligned}$$

(c) Bad notation, but $\{x_i\}, \{y_j\}, \{z_k\}$ denote the possible values of $X, Y$, and $X+Y$. $A = \{(i,j) \mid x_i + y_j = z_k, z_k \in \{z_k\}\}$ and let $B = \{(i,j) \mid x_i \in \{x_i\}, y_i \in \{y_i\}\}$. If $(i,j) \in B$ then by definition $x_i + y_j = z_k$ for some $z_k \in \{z_k\}$ and thus $B \subset A$. If $(i,j) \in A$ then clearly $x_i \in \{x_i\}, y_j \in \{y_j\}$. Therefore $A = B$ and the two ways to write indices are equivalent.

(d)

$$P(X=x_i) = P(X=x_i, Y\in\mathbb{R}) = P(X=x_i, Y\in\{y_j\}) = \sum_{y_i\in\{y_j\}} P(X=x_i, Y=y_j)$$

where the last step is true since the events are pairwise disjoint when we let $y_j$ vary.

(e) This I believe has already been shown in class twice.

$$\begin{aligned} E[X+Y] &= \sum_{x_i\in\{x_i\}} \sum_{y_j\in\{y_j\}} (x_i+y_j)P(X=x_i, Y=y_j) \\ &= \sum_{x_i\in\{x_i\}} \sum_{y_j\in\{y_j\}} x_i\cdot P(X=x_i, Y=y_j) + \sum_{x_i\in\{x_i\}} \sum_{y_j\in\{y_j\}} y_j\cdot P(X=x_i, Y=y_j) \\ &= \sum_{x_i\in\{x_i\}} x_i \sum_{y_i\in\{y_j\}} P(X=x_i, Y=y_j) + \sum_{y_j\in\{y_j\}} \sum_{x_i\in\{x_i\}} y_j\cdot P(X=x_i, Y=y_j) \\ &= \sum_{x_i\in\{x_i\}} x_i P(X=x_i) + \sum_{y_j\in\{y_j\}} y_j \sum_{x_i\in\{x_i\}} P(X=x_i, Y=y_j) \\ &= \sum_{x_i\in\{x_i\}} x_i P(X=x_i) + \sum_{y_j\in\{y_j\}} y_j P(Y=y_j) \\ &= E[X]+E[Y]. \end{aligned}$$