MATH 407 Problem Set 7

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Problem 4.46 For 3 examiners, the student needs two passes out of three. For the 1/3 probability for an on day, the probability that he/she passes is

$$P(2 \text{ passes}) + P(3 \text{ passes}) = 3 \cdot 0.8^2 \cdot 0.2^1 + 0.8^3 = 0.896.$$

For the 2/3 probability for an off day, the probability that he/she passes is

$$P(2 \text{ passes}) + P(3 \text{ passes}) = 3 \cdot 0.4^2 \cdot 0.6 + 0.4^3 = 0.352.$$

Therefore the total probability of passing with 3 examiners is

$$\frac{1}{3} \cdot 0.896 + \frac{2}{3} \cdot 0.352 \approx 0.533.$$

For 5 examiners, the student needs three passes out of 5. For the 1/3 probability for an on day, the probability that he/she passes is

$$P(3) + P(4) + P(5) = C_5^2 \cdot 0.8^3 \cdot 0.2^2 + 5 \cdot 0.8^4 \cdot 0.2 + 0.8^5 = 0.9428,$$

and for the 2/3 probability for an off day, the probability that he/she passes is

$$P(3) + P(4) + P(5) = C_5^2 \cdot 0.4^3 \cdot 0.6^2 + 5 \cdot 0.4^4 \cdot 0.6 + 0.4^5 \approx 0.3174.$$

Therefore the total probability of passing with 5 examiners is

$$\frac{1}{3} \cdot 0.9428 + \frac{2}{3} \cdot 0.3174 \approx 0.5259.$$

It follows that this student should choose the one with 3 examiners.

Theoretical Ex 4.4 (a) Notice that

$$\frac{4}{n(n+1)(n+2)} = \frac{2}{n} - \frac{4}{n+1} + \frac{2}{n+2}.$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} P(X = n) &= \sum_{n=1}^{\infty} \left[\frac{2}{n} - \frac{4}{n+1} + \frac{2}{n+2}\right] \\
&= \sum_{n=1}^{\infty} \frac{2}{n} - \sum_{n=2}^{\infty} \frac{4}{n} + \sum_{n=3}^{\infty} \frac{2}{n} \\
&= 2 + 1 - 2 + \sum_{n=3}^{\infty} \frac{2}{n} + \sum_{n=3}^{\infty} \frac{2}{n} - \sum_{n=2}^{\infty} \frac{4}{n} \\
&= 1.
\end{aligned}$$

(b) More computations:

$$E[X] = \sum_{n=1}^{\infty} \frac{4n}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{4}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \left[\frac{4}{n+1} - \frac{4}{n+2}\right] = \frac{4}{2} = 2.$$

(c) Even more computations:

$$E[X^2] = \sum_{n=1}^{\infty} \frac{4n^2}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{4n}{(n+1)(n+2)}.$$

Let a_n denote the n^{th} term of the series above. Applying the ratio test gives

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \limsup_{n\to\infty} \frac{4(n+1)(n+1)(n+2)}{4n(n+2)(n+3)} = \limsup_{n\to\infty} \frac{(n+1)^2}{n(n+3)} = 1$$

so the series diverges and $E[X^2] = \infty$.

Theoretical Ex 4.20

$$\begin{aligned}
E[X^n] &= \sum_{x=0}^{\infty} x^n P(x) = \sum_{x=0}^{\infty} x^n e^{-\lambda} \frac{\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} x^n e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{x=1}^{\infty} x^{n-1} e^{-\lambda} \frac{\lambda^x}{(x-1)!} \\
&= \lambda \sum_{x=1}^{\infty} x^{n-1} e^{-\lambda} \frac{\lambda^{x-1}}{(x-1)!} \\
&= \lambda \sum_{x=0}^{\infty} (x+1)^{n-1} e^{-\lambda} \frac{\lambda^x}{x!} \\
&= \lambda E[(X+1)^{n-1}].
\end{aligned}$$

Therefore,

$$\begin{aligned}
E[X^3] &= \lambda E[(X+1)^2] \\
&= \lambda(E[X^2] + 2E[X] + 1) \\
&= \lambda(\lambda E[(X+1)] + 2E[X] + 1) \\
&= \lambda(\lambda(\lambda+1) + 2\lambda + 1) \\
&= \lambda^3 + 3\lambda^2 + \lambda.
\end{aligned}$$

Self-test 4.20 Let X be a geometric random variable with parameter p, i.e., $P(X = x) = (1-p)^{x-1}p$. Then,

$$\begin{aligned}
E[1/X] &= \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}p}{n} = \frac{p}{1-p} \sum_{n=1}^{\infty} \frac{(1-p)^n}{n} \\
&= \frac{p}{1-p} \sum_{n=1}^{\infty} \left[\int_0^{1-p} \tilde{x}^{n-1}\, \mathrm{d}\tilde{x}\right] \\
&= \frac{p}{1-p} \int_0^{1-p} \sum_{n=1}^{\infty} \tilde{x}^{n-1}\, \mathrm{d}\tilde{x} && \text{(Interchangeable } \textstyle\sum \text{ and } \int \text{ by uniform convergence)} \\
&= \frac{p}{1-p} \int_0^{1-p} \frac{1}{1-\tilde{x}}\, \mathrm{d}\tilde{x} \\
&= \frac{p}{1-p} \Big[-\log(\tilde{x})\Big]_{\tilde{x}=0}^{1-p} \\
&= \frac{-p\log(p)}{1-p}.
\end{aligned}$$