MATH 407 Homework 8

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Ex.5.2 First notice that

$$\int xe^{-x/2}\,\mathrm{d}x = -2xe^{-x/2} + \int 2e^{-x/2}\,\mathrm{d}x = -2xe^{-x/2} - 4e^{-x/2}.$$

Therefore if $\int_0^\infty Cxe^{-x/2}\,\mathrm{d}x = 1$ we have $4c = 1 \implies c = 1/4$. Thus

$$P(X \geqslant 5) = \frac{1}{4}\int_5^\infty xe^{-x/2}\,\mathrm{d}x = \frac{-2xe^{-x/2} - 4e^{-x/2}}{4}\Big|_{x=5}^{\infty} = \frac{10e^{-5/2} + 4e^{-5/2}}{4} = \frac{8e^{-5/2}}{4}.$$

Ex.5.6 (a) As shown above, this is indeed a density function. Then

$$\begin{aligned} E[X] &= \frac{1}{4}\int_0^\infty x^2 e^{-x/2}\,\mathrm{d}x \\ &= \frac{1}{4}\left[-2x^2e^{-x/2}\right]_{x=0}^{\infty} + \frac{1}{4}\int_0^\infty 4xe^{-x/2}\,\mathrm{d}x \\ &= \frac{1}{4}\cdot 16 = 4. \end{aligned}$$

(b) Simply notice that $f(x) = f(-x)$. This means the integral evaluating $E[X]$ evaluates to 0 and so $E[X] = 0$.

(c)

$$E[X] = \int_5^\infty \frac{5x}{x^2}\,\mathrm{d}x = \infty.$$

Ex.5.38 Assume X is uniformly distributed over $(-1, 1)$.

(a) $P(|X| > 1/2) = P(X \leqslant -0.5) + P(X > 0.5) = 1/2$.

(b) $F_{|X|}(x) = P(|X| \leqslant x) = P(-x \leqslant X \leqslant x) = 2x/2 = x$. Therefore taking the derivative gives $f_{|X|}(x) = 1$. To put formally,

$$f_{|X|}(x) = \begin{cases} 1 & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

Theoretical Ex.5.8 Since

$$E[X^2] = \int_0^c x^2 P(x)\,\mathrm{d}x \leqslant \int_0^c cxP(x)\,\mathrm{d}x = cE[x],$$

we have

$$\begin{aligned} \mathrm{Var}[X] &= E[X^2] - E[X]^2 \leqslant E[cX] - E[X]^2 \\ &\leqslant cE[X] - E[X]^2 = E[X](c - E[X]) \\ &= c^2(E[X]/c)(1 - E[X]/c) \leqslant \frac{c^2}{4}. \end{aligned}$$

Theoretical Ex.5.11 (a) Recall that here $f_Z(x) = \exp(-x^2/2)/\sqrt{2\pi}$. Then (assuming lim and $\int$ are interchangeable in this example)

$$\begin{aligned}
E[g'(Z)] &= \int_{-\infty}^{\infty} g'(z) f_Z(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} \frac{\exp(-z^2/2)}{\sqrt{2\pi}} g'(z)\,\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} g'(z)\,\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{h\to 0} \frac{g(z+h)-g(z)}{h} e^{-z^2/2}\,\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{h\to 0} \frac{g(z+h)}{h} e^{-z^2/2}\,\mathrm{d}z - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{h\to 0} \frac{g(z)}{h} e^{-z^2/2}\,\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{h\to 0} \frac{g(z)}{h} e^{-(z-h)^2/2}\,\mathrm{d}z - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{h\to 0} \frac{g(z)}{h} e^{-z^2/2}\,\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(z) \lim_{h\to 0} \frac{\exp(-(z-h)^2/2) - \exp(-z^2/2)}{h}\,\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(z) \left[z \cdot \exp(-z^2/2)\right]\mathrm{d}z \\
&= \int_{-\infty}^{\infty} z g(z) f_Z(z)\,\mathrm{d}z = E[Zg(Z)].
\end{aligned}$$

(b) It follows from (a) that, if we let $g(Z) := Z^n$ then $g'(Z) = nZ^{n-1}$ and thus

$$nE[Z^{n-1}] = E[nZ^{n-1}] = E[Z \cdot Z^n] = E[Z^{n+1}].$$

(c) Simple. From above we have

$$E[Z^4] = 3E[Z^2] = 3(1 \cdot E[Z^0]) = 3.$$