Math 407 Homework 9

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Ex.5.16 Suppose $X \sim N(\mu, \sigma^2)$. Immediately we see that $P(X < 1.8 \cdot 10^5) = P(X > 3.2 \cdot 10^5)$ so the mean must be precisely the midpoint, i.e., $\mu = 2.5 \cdot 10^5$. Now we look up the value $1 - 0.25 = 0.75$ in the distribution table and find that, for a standard normal, $P(X > 0.675) \approx 0.75$. Therefore the standard distribution is approximately $(3.2 \cdot 10^5 - 2.5 \cdot 10^5)/0.675 = 1.04 \cdot 10^5$.

(a) $X < 2 \cdot 10^5$ corresponds to $Z < (2 \cdot 10^5 - 2.5 \cdot 10^5)/(1.04 \cdot 10^5) \implies Z < -0.482$. Thus

$$P(Z < -0.482) = P(Z > 0.482) = 1 - P(Z < 0.482) \approx 1 - 0.68439 = 0.31561.$$

(b) $2.8 \cdot 10^5 < X < 3.2 \cdot 10^5$ corresponds to $0.3/1.04 < Z < 0.7/1.04$. Thus

$$P(0.289 < Z < 0.675) = \Phi(0.675) - \Phi(0.289) \approx 0.75 - 0.61 = 0.14.$$

Ex.5.21 $\sigma = 2.5$. $6'2$ is 74 inches, so it's $3/2.5 = 1.2\sigma$ more than the mean. So $P(Z > 1.2) = 1 - P(Z < 1.2) \approx 0.11507$. $6'5$ is 77 inches, $6/2.5 = 2.4\sigma$ more than the mean. Then $P(Z > 2.4) = 1 - P(Z < 2.4) \approx 0.0082$. Conditioning it over the percentage of the 6-footer club members (which is $1/2.5 = 0.4\sigma$ above mean) we get

$$P(Z > 2.4 \mid Z > 0.4) = \frac{P(Z > 2.4 \wedge Z > 0.4)}{P(Z > 0.4)} = \frac{0.0082}{1 - 0.6554} = 0.024.$$

T.Ex.5.15 The cdf of cX, should it be well-defined, is

$$F_{xC}(x) = P(\{cX < x\}) = P(\{X < x/c\}) = 1 - e^{-\lambda x/c}$$

which indeed is the cdf of the exponential r.v. with parameter λ/c.

S.T. 5.23 (a) Clear enough, $f(x) > 0$ for all x as $\exp(\cdot)$ is always positive. It remains to check that the integral of $f(x)$ evaluates to 1:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x &= \int_{-\infty}^{0} \frac{e^x}{3}\,\mathrm{d}x + \int_0^1 \frac{1}{3}\,\mathrm{d}x + \int_1^{\infty} \frac{e^{-x-1}}{3}\,\mathrm{d}x \\ &= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.\end{aligned}$$

(b) Simple.

$$\begin{aligned}E[X] &= \int_{-\infty}^{\infty} xf(x)\,\mathrm{d}x = \int_{-\infty}^{0} \frac{xe^x}{3}\,\mathrm{d}x + \int_0^1 \frac{x}{3}\,\mathrm{d}x + \int_1^{\infty} \frac{xe^{-x-1}}{3}\,\mathrm{d}x \\ &= \frac{xe^x}{3}\bigg|_{x=-\infty}^{0} - \int_{-\infty}^{0} \frac{e^x}{3}\,\mathrm{d}x + \frac{x^2}{6}\bigg|_{x=0}^{1} + \frac{-xe^{-x-1}}{3}\bigg|_{x=1}^{\infty} - \int_1^{\infty} \frac{-e^{-x-1}}{3}\,\mathrm{d}x \\ &= 0 - \frac{1}{3} + \frac{1}{6} + \frac{2}{3e^2} = -\frac{1}{6} + \frac{2}{3e^2}.\end{aligned}$$

S.T.5.24 Given $\theta > 0$, it is clear that $f(x) > 0$ for all $x > 0$ as $\exp(\cdot) > 0$ as well. It remains to check $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}\,\mathrm{d}x \\
&= \frac{\theta^2}{1+\theta}\int_0^{\infty}(1+x)e^{-\theta x}\,\mathrm{d}x \\
&= \frac{\theta^2}{1+\theta}\left[\int_0^{\infty} e^{-\theta x}\,\mathrm{d}x + \int_0^{\infty} xe^{-\theta x}\,\mathrm{d}x\right] \\
&= \frac{\theta^2}{1+\theta}\left[-\frac{e^{-\theta x}}{\theta}\bigg|_{x=0}^{\infty} - \frac{xe^{-\theta x}}{\theta}\bigg|_{x=0}^{\infty} - \int_0^{\infty}\frac{-e^{-\theta x}}{\theta}\,\mathrm{d}x\right] \\
&= \frac{\theta^2}{1+\theta}\left[\frac{1}{\theta} + \frac{e^{-\theta x}}{\theta^2}\bigg|_{x=0}^{\infty}\right] \\
&= \frac{\theta^2}{1+\theta}\left[\frac{1}{\theta} + \frac{1}{\theta^2}\right] = 1.
\end{aligned}$$

For the mean,

$$\begin{aligned}
E[X] &= \frac{\theta^2}{1+\theta}\int_0^{\infty} x(1+x)e^{-\theta x}\,\mathrm{d}x \\
&= \frac{\theta^2}{1+\theta}\left[\int_0^{\infty} xe^{-\theta x}\,\mathrm{d}x + \int_0^{\infty} x^2e^{-\theta x}\,\mathrm{d}x\right] \\
&= \frac{\theta^2}{1+\theta}\left[\frac{1}{\theta^2} - \frac{x^2e^{-\theta x}}{\theta}\bigg|_{x=0}^{\infty} - \int_0^{\infty}\frac{-2xe^{-\theta x}}{\theta}\,\mathrm{d}x\right] \\
&= \frac{\theta^2}{1+\theta}\left[\frac{1}{\theta^2} + \frac{2}{\theta}\int_0^{\infty} xe^{-\theta x}\,\mathrm{d}x\right] = \frac{\theta^2}{1+\theta}\left[\frac{1}{\theta^2} + \frac{2}{\theta^3}\right] \\
&= \frac{\theta+2}{\theta(\theta+1)}.
\end{aligned}$$

Similarly, we first compute $E[X^2]$ before computing the variance:

$$\begin{aligned}
E[X^2] &= \frac{\theta^2}{1+\theta}\left[\int_0^{\infty} x^2e^{-\theta x}\,\mathrm{d}x + \int_0^{\infty} x^3e^{-\theta x}\,\mathrm{d}x\right] \\
&= \frac{\theta^2}{1+\theta}\left[\frac{2}{\theta^3} - \frac{x^3e^{-\theta x}}{\theta}\bigg|_{x=0}^{\infty} - \int_0^{\infty}\frac{-3x^2e^{-\theta x}}{\theta}\,\mathrm{d}x\right] \\
&= \frac{\theta^2}{1+\theta}\left[\frac{2}{\theta^3} + \frac{3}{\theta}\int_0^{\infty} x^2e^{-\theta x}\,\mathrm{d}x\right] = \frac{\theta^2}{1+\theta}\left[\frac{2}{\theta^3} + \frac{6}{\theta^4}\right] \\
&= \frac{2(\theta+3)}{\theta^2(\theta+1)}.
\end{aligned}$$

Therefore

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{2(\theta+3)}{\theta^2(\theta+1)} - \frac{(\theta+2)^2}{\theta^2(\theta+1)^2} = \frac{2(\theta+2)^2 - 2 - (\theta+2)^2}{\theta^2(\theta+1)^2} = \frac{(\theta+2+\sqrt{2})(\theta+2-\sqrt{2})}{\theta^2(\theta+1)^2}.$$