

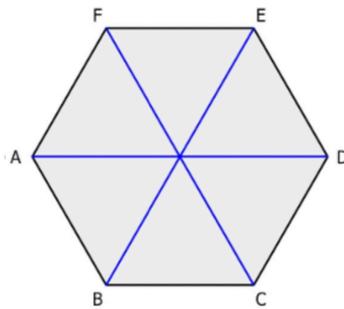
MATH 410 Final Exam

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Problem 1

How many ways are there to paint a hexagonal table in triangular sections with three colors of paint, assuming the bottom of the table remains unpainted?



Solution. Two patterns are considered essentially the same if one can be obtained by rotating another. Therefore the group acting on this table is R_6 . Now we invoke Burnside's lemma:

Action	Elements in R_6	$ \text{Fix}(\sigma) $	Total
Rotation by 60°	r, r^5	3	6
Rotation by 120°	r^2, r^4	$3^2 = 9$	18
Rotation by 180°	r^3	$3^3 = 27$	27
Nothing	e	$3^6 = 729$	729

Therefore the total number of essentially different color patterns is

$$\#\text{Orbits} = \sum_{\sigma \in R_6} |\text{Fix}(\sigma)| = \frac{6 + 18 + 27 + 729}{6} = 130.$$

Problem 2

Let G be an arbitrary group of order 145.

(i) How many subgroups of order 5 does G have?

Since $145 = 5 \cdot 29$, by Sylow III, $N_5 \equiv 1 \pmod{5}$ and $N_5 \mid 29$, which gives the only possibility of $N_5 = 1$.

- (ii) How many elements of order 5 does G have?

Only four, the non-identity elements in that subgroup of order 5. Otherwise if $o(g) = 5$ for some other g then $\langle g \rangle$ is also a subgroup of order 5, contradicting the uniqueness as stated above.

- (iii) How many elements of order 145 does G have?

We first show that G is cyclic. By the same token we can show there's only one subgroup of order 29. Let $\langle x \rangle$ be the subgroup of order 5 and let $\langle y \rangle$ be that of order 29. Then $\langle x \rangle \cap \langle y \rangle = \{e\}$ as the intersection of two normal subgroups must be normal. Clearly $xyy^{-1}x^{-1} = e$: on the other hand

$$xyx^{-1}y^{-1} = \begin{cases} (xyx^{-1})y^{-1} \in \langle y \rangle y^{-1} = \langle y \rangle \\ x(yx^{-1}y^{-1}) \in x \langle x \rangle = \langle x \rangle \end{cases}$$

and this suggests $xyx^{-1}y^{-1} = e$. Thus $x^{-1}y^{-1} = y^{-1}x^{-1}$ and so $xy = yx$. It follows that $o(xy) = 5, 29 = 145$ and this shows G is cyclic. Thus the number of elements of order 145 is $\varphi(145) = 112$.

Problem 3

Let $G = S_4$ and let G act on itself by conjugation.

- (i) Find $\text{Orb}((34))$.

Recall that $\sigma(34)\sigma^{-1} = (\sigma(3), \sigma(4))$ so its orbit consists of all the 2-cycles, i.e.,

$$\text{Orb}((34)) = \{(12), (13), (14), (23), (24), (34)\}.$$

- (ii) Calculate $|\text{Stab}((34))|$.

By the Orbit-Stabilizer theorem, $|\text{Stab}((34))| = |S_4|/|\text{Orb}((34))| = 4$.

- (iii) Find $\text{Stab}((34))$.

If $\sigma(3) = 3, \sigma(4) = 4$ we have two options: e or (12) . Otherwise we have also two options: (34) or $(12)(34)$. They together constitute $\text{Stab}((34))$.

Problem 4

Let $G := \text{GL}_2(\mathbb{Z}/3\mathbb{Z})$ and H be the subgroup generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

- (i) What is the order of G ?

The order of G is $(3^2 - 1)(3^2 - 3) = 48$ as shown in one of the early lectures. Basically the first $3^2 - 1$ comes from the fact that the first row of G can be any nonzero (row) vector, whereas once that's determined, the second row can be anything but a multiple of the first row, hence the -3 .

- (ii) What is the order of G/H ?

Clearly H has order 3 as $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ and $|\mathbb{Z}/3\mathbb{Z}| = 3$. Thus $|G/H| = |G|/|H| = 16$.

(iii) Is G/H a group?

No, because H is not normal:

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

and left-multiplying an element of H by an element of G gives something not in H .

Problem 5

Let R be the ring $R = \mathbb{Z}/3\mathbb{Z}[x]$ and I the ideal $I = \langle x^3 + 1 \rangle$.

(i) Describe the elements in R/I .

Note that in R/I $[x^3 + 1] = [0]$ so $[x^3] = [-1] = [2]$. This means any polynomial of order ≥ 3 can be reduced to another polynomial of order < 3 . Therefore

$$R/I = \{[ax^2 + bx + c] : a, b, c \in \mathbb{Z}/3\mathbb{Z}\}.$$

(ii) Is R/I a field? An integral domain?

Neither. Notice that $[x + 1][x^2 - x + 1] = [x^3 + 1] = [0]$.

Problem 6

The Cayley table for a group G is given below.

	1	a	a ²	a ³	a ⁴	a ⁵	b	ab	a ² b	a ³ b	a ⁴ b	a ⁵ b
1	1	a	a ²	a ³	a ⁴	a ⁵	b	ab	a ² b	a ³ b	a ⁴ b	a ⁵ b
a	a	a ²	a ³	a ⁴	a ⁵	1	ab	a ² b	a ³ b	a ⁴ b	a ⁵ b	b
a ²	a ²	a ³	a ⁴	a ⁵	1	a	a ² b	a ³ b	a ⁴ b	a ⁵ b	b	ab
a ³	a ³	a ⁴	a ⁵	1	a	a ²	a ³ b	a ⁴ b	a ⁵ b	b	ab	a ² b
a ⁴	a ⁴	a ⁵	1	a	a ²	a ³	a ⁴ b	a ⁵ b	b	ab	a ² b	a ³ b
a ⁵	a ⁵	1	a	a ²	a ³	a ⁴	a ⁵ b	b	ab	a ² b	a ³ b	a ⁴ b
b	b	a ⁵ b	a ⁴ b	a ³ b	a ² b	ab	a ³	a ²	a	1	a ⁵	a ⁴
ab	ab	b	a ⁵ b	a ⁴ b	a ³ b	a ² b	a ⁴	a ³	a ²	a	1	a ⁵
a ² b	a ² b	ab	b	a ⁵ b	a ⁴ b	a ³ b	a ⁵	a ⁴	a ³	a ²	a	1
a ³ b	a ³ b	a ² b	ab	b	a ⁵ b	a ⁴ b	1	a ⁵	a ⁴	a ³	a ²	a
a ⁴ b	a ⁴ b	a ³ b	a ² b	ab	b	a ⁵ b	a	1	a ⁵	a ⁴	a ³	a ²
a ⁵ b	a ⁵ b	a ⁴ b	a ³ b	a ² b	ab	b	a ²	a	1	a ⁵	a ⁴	a ³

(i) How many elements of order 4 does G have?

Notice that $b^4 = (b^2)^2 = (a^3)^2 = e$ and the same holds for all elements of form $a^n b$. However, a^3 is the only element of order 2 and all powers of a fail to satisfy $(a^n)^2 = a^3$. Therefore there are only 6 elements of order 4, namely b, ab, a^2b, \dots, a^5b .

(ii) Let G act on itself by conjugation. What is $\text{Stab}(b)$?

OK this is weird... Clearly from the table $a^n b a^{6-n} = b$ (check the topright "diagonal" consisting of b 's). In addition, for elements of form $a^n b$,

$$(a^n b)b(a^n b)^{-1} = a^n b b (a^{n+3} b) = a^n a^3 a^{n+3} b = a^{2n+6} b,$$

and we see b and $a^3 b$ are also stabilizers of b . Hence

$$\text{Stab}(b) = \{e, a, \dots, a^5, b, a^3 b\}.$$

(But this violates the Orbit-Stabilizer theorem, or did I make a mistake?)

Problem 7

Let G be a group having the presentation $\langle r, s \mid r^5 = s^2 = 1, rs = r^2 \rangle$. What is the order of G ?

If $rs = r^2$ then left-multiplying by r^4 gives $r^4 rs = r^4 r^2 \implies s = r$. This means $r^5 = r^2 \implies r^3 = 1$. Then division algorithm eventually gives $r = 1$ [!] This means $G = \{1\}$ (or $\{e\}$ which I prefer).