

# MATH 410 Homework 5

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3.2.4 If  $G, G'$  are groups and  $G \cong G'$  then there exists a bijection  $T : G \rightarrow G'$  which implies  $|G| = |G'|$ .  $\square$

3.2.12 We have already shown this in class. Let  $C_g : G \rightarrow G$  be defined as  $x \mapsto g^{-1}xg$ . Then  $C_g$  is injective since

$$g^{-1}xg = g^{-1}yg \implies g(g^{-1}xg)g^{-1} = g(g^{-1}yg)g^{-1} \implies x = y.$$

$C_g$  is surjective because, for all  $x \in G$ , we have

$$C_g(gxg^{-1}) = g^{-1}(gxg^{-1})g = x.$$

Finally,  $C_g$  preserves group structure because

$$C_g(xy) = g^{-1}(xy)g^{-1} = (g^{-1}gx)(g^{-1}yg) = C_g(x)C_g(y).$$

Hence  $C_g$  is an isomorphism.  $\square$

3.2.13 Reflexivity is immediate as  $a = e^{-1}ae$ . Symmetry is also clear since if  $x = g^{-1}yg$  then  $y = gxg^{-1} = (g^{-1})^{-1}xg^{-1}$ . Finally, for transitivity, if  $x = g^{-1}yg$  and  $y = h^{-1}zh$  for some  $g, h \in G$ , then

$$x = g^{-1}(h^{-1}zh)g = (hg)^{-1}z(hg).$$

By definition, the equivalence class of  $x \in G$  is the set of all elements  $y \in G$  satisfying the relation with  $x$ , so  $[x] = \{y \in G : y = g^{-1}xg \text{ for some } g \in G\}$ . Both  $[x] \subset x^G$  and  $x^G \subset [x]$  are trivial statements that simply follow from definition.  $\square$

3.2.14 By the hint, we have

$$\sigma(a_1 \dots a_n)\sigma^{-1} = (\sigma(a_1) \dots \sigma(a_n)).$$

Notice that this holds true for the following version as well:

$$\sigma(a_1 a_2)(a_3 a_4)\sigma^{-1} = (\sigma(a_1)\sigma(a_2))(\sigma(a_3)\sigma(a_4)).$$

This means that, given an element in  $A_4$ , all other elements in its conjugacy class are of the same structure.

One trivial example is the conjugacy of  $e$ , namely  $\{e\}$  itself.

Now consider  $(12)(34)$ . Its conjugacy class should contain all other products of two disjoint two-cycles, namely  $\{(12)(34), (13)(24), (14)(23)\}$ .

As another example, consider  $(12)(13) = (123)$ . Conjugating it by  $(13)(14) = (134)$  gives  $(243)$ . Similarly, conjugating it by  $(23)(24) = (234)$  gives  $(134)$  and conjugating it by  $(13)(34) = (143)$  gives  $(142)$ . The remaining 4 elements form another conjugacy class,  $\{(132), (234), (143), (124)\}$ .

3.2.18 Define  $T : \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \rightarrow \mathbb{Z}/n\mathbb{Z}^\times$  by  $T(\sigma) = \sigma(1)$ . Notice that  $\sigma(1)$  must be a generator of  $\mathbb{Z}/n\mathbb{Z}$  as it's assumed to be an automorphism, which means

$$\{\sigma(1), \sigma(2), \dots, \sigma(n-1), \sigma(n)\} = \{\sigma(1), 2\sigma(1), \dots, (n-1)\sigma(1), n\sigma(1)\} = \mathbb{Z}/n\mathbb{Z}.$$

Therefore  $T$  indeed maps  $\text{Aut}(\mathbb{Z}/n\mathbb{Z})$  to  $\mathbb{Z}/n\mathbb{Z}^\times$ . To show  $T$  is injective, if  $\sigma_1(1) = \sigma_2(1)$  then  $\sigma_1(k) = k\sigma_1(1) = k\sigma_2(1) = \sigma_2(k)$  for all  $k$ , namely  $\sigma_1 = \sigma_2$ . To show  $T$  is surjective, take any  $m \in \mathbb{Z}/n\mathbb{Z}^\times$ . We simply need to define  $\sigma_m$  to be the automorphism such that  $\sigma_m(1) = m$ . It remains to show  $T$  preserves group structure:

$$(\sigma_1 \circ \sigma_2)(1) = \sigma_1(\sigma_2(1)) = \sigma_1(\sigma_2(1) \cdot 1) = \sigma_2(1)\sigma_1(1) = \sigma_1(1)\sigma_2(1).$$

Therefore  $T$  is an isomorphism and  $\text{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}^\times$ .

3.2.19 Consider  $\log : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  defined by

$$\log(x) := \int_1^x \frac{1}{t} d\tilde{t}.$$

As the integrand  $1/\tilde{t}$  is always positive,  $\log$  is clearly injective. It preserves group structure because

$$\begin{aligned} \log(xy) &= \int_1^{xy} \frac{1}{t} d\tilde{t} = \int_1^x \frac{1}{t} d\tilde{t} + \int_x^{xy} \frac{1}{t} d\tilde{t} \\ &= \log(x) + \int_1^y \frac{1}{\tilde{u}} d\tilde{u} \\ &= \log(x) + \log(y). \end{aligned}$$

To show it is onto  $(-\infty, \infty)$ , notice that

$$\log(2^n) = n \log 2 = \int_1^{2^n} \frac{1}{t} d\tilde{t} \geq \frac{n}{2}$$

so the  $\log$  function is unbounded from above. Also, since  $\log(2^n 2^{-n}) = 0 = \log(2^n) + \log(2^{-n})$ , the  $\log$  function is unbounded from below. By FTC, the continuity of  $1/\tilde{t}$  for  $t > 0$  implies the continuity of  $\log$ , and by IMT and the unboundedness,  $\log$  must surject onto  $\mathbb{R}$ .  $\square$

3.3.2  $\varphi(20) = 8$  and  $\mathbb{Z}/20\mathbb{Z}^\times = \{1, 3, 7, 9, 11, 13, 17, 19\}$ . Time to enumerate by brute force:

$$[1]H = \{1, 11\} = H$$

$$[3]H = \{3, 13\}$$

$$[7]H = \{7, 17\}$$

$$[9]H = \{9, 19\}.$$

No need to enumerate 11, 13, 17, 19 since they have already appeared above. Thus

$$(\mathbb{Z}/20\mathbb{Z}^\times)/H = \{\{1, 11\}, \{3, 13\}, \{7, 17\}, \{9, 19\}\}.$$

3.3.4 For  $\implies$ , suppose  $H \subset G$  is normal. The left multiplying  $g$  on both sides of  $g^{-1}Hg = H$  gives  $Hg = gH$ .

For  $\impliedby$ , suppose  $Hg = gH$ . We want to show  $g^{-1}Hg = H$  for all  $g \in G$ . Again, pick  $g \in G$  and fix it. Consider  $\{g^{-1}hg : h \in H\}$  and  $H$ . We first show  $\subset$ . Pick  $g^{-1}hg \in g^{-1}Hg$  (which is equivalent to picking  $h \in H$ ). Since  $gH = Hg$ ,  $gh = \tilde{h}g$  for some  $\tilde{h} \in H$ . Then,  $g^{-1}\tilde{h}g = g^{-1}gh = h$ , as desired. Now for  $\supset$ , pick  $h \in H$  and consider  $g^{-1}hg$ . Once again, since  $gH = Hg$ ,  $gh = \tilde{h}g$  for some  $\tilde{h} \in H$ . Then  $h = g^{-1}gh = g\tilde{h}g \in g^{-1}Hg$ . Thus  $g^{-1}Hg = H$ .  $\square$

3.3.7 Suppose  $|G/H| = 2$ . Pick  $g \in G$ . If  $g \in H$  then  $gH = H = Hg$  by the closure of group. Since  $|G/H| = 2$ , there will be precisely two left cosets (and two right cosets) of the same size that partition  $G$ . Therefore, if  $g \notin H$  then  $gH$  must be the other left coset and  $Hg$  the other right coset, satisfying

$$H \sqcup gH = G = H \sqcup Hg \text{ (disjoint).}$$

Therefore  $gH = Hg$  in either case, and by Exercise 3.3.4 we conclude that  $H$  must be normal. □

3.3.13 Simply consider the trivial subgroup consisting only of  $I_{2 \times 2}$ .

 End of Homework 5 