MATH 410 Midterm 1: Takehome Component

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Problem 1

Let $G = GL(2, \mathbb{Z}/7\mathbb{Z})$ and $H \subset G$ be matrices of form

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \,\middle|\, ac \not\equiv 0 \pmod 7 \right\}.$$

(i)   Show that H is a subgroup of G.

> *Proof.* We use the two-step method. Let an arbitrary $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ be given. Its inverse
>
> $$\frac{1}{[ac]} \begin{bmatrix} c & [b]^{-1} \\ 0 & a \end{bmatrix}$$
>
> is also in H since
>
> $$\left(c[ac]^{-1}\right)\left(a[ac]^{-1}\right) = ac\left([ac]^{-1}\right)^2 = [ac]^{-1} \neq 0.$$
>
> (Here x^{-1} denotes the additive inverse of $x \in \mathbb{Z}/7\mathbb{Z}$, namely $7 - x$, and $1/[x]$ denotes the multiplicative inverse of x mod 7.) It is easy to verify that
>
> $$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \frac{1}{[ac]} \begin{bmatrix} c & 7-b \\ 0 & a \end{bmatrix} = \frac{1}{[ac]} \begin{bmatrix} ac & 7a \\ 0 & ac \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
>
> For closure, simply notice that $\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & * \\ 0 & c_1 c_2 \end{bmatrix}$. Since $a_1 c_1, a_2 c_2 \not\equiv 0 \pmod 7$, neither does their product $(a_1 a_2)(c_1 c_2)$ and hence closure follows. Indeed H is a subgroup of G. $\qquad\square$

(ii)   Define a relation on $A, B \in G$ by $A \sim B \iff A = BC$ for some $C \in H$. Show that $\sim$ is an equivalence relation on G.

> *Proof.* We verify the criteria of an equivalence relation one by one.
>
> (1) Reflexivity: clearly $I \in H$ and $A = AI$ for all $A \in G$.
>
> (2) Symmetry: if $A \sim B$, i.e., for some $C \in H$ we have $A = BC$, then immediately $B = AC^{-1}$ and, since H is a group, we know $C^{-1} \in H$. Thus $a \sim b \implies b \sim a$. The converse $\impliedby$ is analogous.
>
> (3) Transitivity: if $A \sim B$ and $B \sim C$ then there exist $H_1, H_2 \in H$ with $A = BH_1$ and $B = CH_2$. Hence $A = (CH_2)H_1 = C(H_2 H_1)$ and by closure of H, $H_2 H_1 \in H \implies A \sim C$. $\qquad\square$

Problem 2

Let $H \subset G := SL(2, \mathbb{R})$ be matrices of form

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| ab = 1 \right\}.$$

For any matrix $B \in G$, let

$$BHB^{-1} := \{ BKB^{-1} \mid K \in H \},$$

and finally define $P \subset G$ to be matrices

$$P := \{ B \in G \mid BHB^{-1} = H \}.$$

(i)   Show that P is a subgroup of G.

Proof. First of all, it is very easy to verify H itself is a subgroup of G (we use the one-step method here):

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}^{-1} = \frac{1}{cd} \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix} \implies \frac{1}{cd}(acbd) = 1 \implies H \text{ is closed.}$$

Now take any $M \in P$. It follows that $MHM^{-1} = H$, i.e., $K \in H$ if and only if $MKM^{-1} \in MHM^{-1}$. Since H is a group, $K \in H$ if and only if $K^{-1} \in H$. Then

$$MK^{-1}M^{-1} \in H \iff K^{-1} \in H \iff K \in H \iff MKM^{-1} \in MHM^{-1}.$$

Notice that the first term is nothing but $(MKM^{-1})^{-1}$. Therefore if $M \in P$, so is M^{-1}.

Now, for closure, take $M_1, M_2 \in P$. By assumption $M_1 H M_1^{-1} = H = M_2 H M_2^{-1}$. Therefore,

$$H = M_1 H M_1^{-1} = M_1 \left[M_2 H M_2^{-1} \right] M_1^{-1} = (M_1 M_2) H (M_1 M_2)^{-1},$$

which shows $M_1 M_2 \in P$ and thus P is a subgroup of G. $\square$

(ii)   Determine the elements in P.

> **Solution**
>
> A priori we know $A \in P \implies A \in G$. Let $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then
>
> $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
>
> Let $K := \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in H$, and we have
>
> $$AKA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} adx - bcy & ab(y - x) \\ cd(x - y) & ady - bcx \end{bmatrix}$$
>
> Therefore a necessary condition for $A \in P$ is $ab = cd = 0$. On the other hand, $A \in G$ so $ad - bc = 1$.

Therefore either $b = c = 0$ and $ad = 1$ or $a = d = 0$ and $bc = -1$. The former case gives

$$AKA^{-1} = \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} 1/a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

whereas the latter gives

$$AKA^{-1} = \begin{bmatrix} 0 & b \\ -1/b & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} 0 & -b \\ 1/b & 0 \end{bmatrix} = \begin{bmatrix} y & 0 \\ 0 & x \end{bmatrix}.$$

Immediately we see both cases work just fine $(AHA^{-1} = H)$ so the characterization of P is given by

$$P = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \middle| ad = 1 \right\} \cup \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \middle| bc = -1 \right\}.$$

(iii) Determine whether P is abelian.

Solution

P is <u>not</u> abelian. Consider $A := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 0.5 \\ -2 & 0 \end{bmatrix}$. We have

$$AB = \begin{bmatrix} -2 & 0 \\ 0 & -0.5 \end{bmatrix} \text{ but } BA = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix}.$$