MATH 410 Midterm 1

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Problem 1. Let $G = \mathbb{Z}/15\mathbb{Z}^\times$.

(1) Find the order of G.

Solution

$|G| = \varphi(15) = 15 \cdot (2/3) \cdot (4/5) = 8$.

(2) Is G cyclic?

Solution

G is not cyclic; recall that $\mathbb{Z}/n\mathbb{Z}^\times$ is cyclic if and only if $n = 2, 4$, or is of form p^k or $2p^k$ where p is prime.

Problem 2. Let $\sigma = (1324)(5768)$ and $\tau = (1526)(3847)$ be elements of S_8. Let $H = \langle \sigma, \tau \rangle$.

(1) Find the permutation σ^{-1}.

Solution

Since $\sigma = (1324)(5768) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 7 & 8 & 6 & 5 \end{pmatrix}$, its inverse is simply

$$\sigma^{-1} = (1324)^{-1}(5768)^{-1} = (4231)(8675) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 8 & 7 & 5 & 6 \end{pmatrix}.$$

(2) Show that H is not abelian.

Proof. We show that σ and τ don't commute.

$$\begin{aligned}\sigma\tau &= (1324)(5768)(1526)(3847)\\ &= \begin{pmatrix}1&2&3&4&5&6&7&8\\3&4&2&1&7&8&6&5\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\5&6&8&7&2&1&3&4\end{pmatrix}\\ &= \begin{pmatrix}1&2&3&4&5&6&7&8\\8&7&6&5&3&4&1&2\end{pmatrix}\end{aligned}$$

whereas

$$\begin{aligned}\tau\sigma &= (1526)(3847)(1324)(5768)\\ &= \begin{pmatrix}1&\dots\\5&\dots\end{pmatrix}\begin{pmatrix}\dots&5&\dots\\\dots&7&\dots\end{pmatrix}\\ &= \begin{pmatrix}1&\dots\\7&\dots\end{pmatrix}.\end{aligned}$$

Hence H is not abelian. □

(3) What is the minimum order of H?

Solution

Since

$$(\sigma^m\tau^n)^4 = (1324)^{4m}(5768)^{4m}(1526)^{4n}(3847)^{4n} = \mathrm{id}$$

it makes sense to restrict m and n to 0 to 3. Hence $|H| = 16$.

Problem 3. Let $G = S_{10}$.

(1) What is the largest order of an element in G?

Solution

Clearly in cycle notations we want to find cycles with coprime lengths (otherwise we can't afford dividing the product by a nontrivial gcd; that would significantly decrease the order of a permutation). If $\sigma \in S_{10}$ is the product of two disjoint cycles, the best possible scenario is $o(\sigma) = 3 \cdot 7 = 21$. If σ consists of three disjoint cycles, the best outcome is $o(\sigma) = 2 \cdot 3 \cdot 5 = 30$. Four cycles? Nope. Hence the element with largest order has order 30. One example is shown below:

$$\tilde{\sigma} := (12)(345)(6789\text{ten}).$$

(2) Find a subgroup of order 24.

Solution

Notice that $24 = 4!$. Consider the subgroup of S_{10} where the first 4 numbers are permuted but the rest 6 are fixed. For example, let $H \coloneqq \langle (12), (1234) \rangle$.

(3) Find a subgroup of order 48.

Solution

Notice that $48 = 2 \cdot 24$. Now consider $H' = \langle (12), (1234), (89) \rangle$. Every element in H' can have a permutation among the first four elements and another permutation among the 8^{th} and 9^{th} elements. There are 48 possibilities. Indeed, since (12) and (1234) (viewed as elements of S_4) generate S_4 and (89) (viewed as element of S_2) generate S_2, indeed $|H'| = 48$.