MATH 410 Midterm 2

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Problem 1

Let G be a group of order 102.

(1) Show that G is not simple.

(2) Show that G has a subgroup of order 51.

Proof of 1.1. First we re-write 102 as $17 \cdot 6$. By Sylow II, there exists some Sylow-17 subgroup of G. If it is unique then it is normal and we are done since conjugation provides us with the same Sylow-17 subgroup. Now suppose there are N_{17} such groups. It follows that $N_{17} \equiv 1 \pmod{17}$ and of course $N_{17} \mid 6$. Therefore $N_{17} = 1$ and this Sylow-17 subgroup is unique. Hence G is not simple. □

Proof of 1.2. From 1.1 we know that G has a normal subgroup H of order 17. By Cauchy's theorem, since $3 \mid 102$ and 3 is a prime, there exists an element of order 3 and thus a (cyclic) subgroup K of order 3. In particular, notice that H is also cyclic as 17 is a prime. Pick $h \in H$ (a generator of H) and $k \in K$ (a generator of K). Consider HK. It follows that $o(hk)$ in HK must be $\operatorname{lcm}(17, 3) = 51$ so this group must have order at least 51. It would not have order 102 because for all $h \in H$ and $k \in K$, $(hk)^{51} = h^{51}k^{51} = e$. □

Problem 2

Recall that $D_4 = \langle r, f \mid r^4 = 1, f^2 = 1, frf^{-1} = r^3 \rangle$. Let D_4 act on itself by conjugation.

(1) Calculate $\operatorname{Stab}(r)$.

(2) Calculate $|\operatorname{Orb}(r)|$.

(3) Calculate $\operatorname{Orb}(r)$.

Solution

(1) We verify them one by one: $ere^{-1} = r, rrr^{-1} = r^5 = r, r^2r(r^2)^{-1} = r^5 = r, r^3r(r^3)^{-1} = r^5 = r$. Then since $frf^{-1} = r^3$ and $\operatorname{Stab}(r)$ is a subgroup of D_4 we conclude that $\operatorname{Stab}(r) = \{e, r, r^2, r^3\} = \langle r \rangle$.

(2) By Orbit-Stabilizer Theorem, $|\operatorname{Orb}(r)| = |G|/|\operatorname{Stab}(x)| = 8/4 = 2$.

(3) From (1), since $frf^{-1} = r^3$ the orbit must be $\{r, r^3\}$.

Problem 3

Let $G = S_4$. Recall that for any subgroup H of G, $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$.

(1) Find a Sylow-3 subgroup P of G.

(2) Show $N_G(P) \neq G$.

(3) Show $|N_G(P)| = 6$, You may use that $N_G(P)$ is not normal in G.

(4) Find the cosets of $N_G(P)/P$.

Solution

(1) Since $|S_4| = 24 = 3 \cdot 8$ the Sylow-3 subgroup of G must be of order 3. One example: $\langle (123) \rangle = \{(1,2,3,4),(2,3,1,4),(3,1,2,4)\}$.

(2) For example, consider (234) whose inverse is (324). We have $(234)(123)(324) \notin \langle (123) \rangle$.

(3) By Sylow III, $N_3 \equiv 1 \pmod 3$ and since it divides 24, it must be either 1 or 4. Given the hint that $N_G(P)$ is not normal, N_3 cannot be 1 (otherwise we have a unique Syloe-3 subgroup that is normal!). Thus $N_3 = 4$ so $|G/N_G(P)| = 4$. This shows that $|N_G(P)| = 24/4 = 6$.

(4) For example consider $(12)(123)(12)^{-1} = (312)$. It follows that $N_G(P)$ must contain $\langle (312) \rangle$. The cosets $N_G(P)/P$ are of form $(ab)\langle (123) \rangle$.