Beginning of March 22, 2021

Example 0.0.1. The **Klein 4-group**, $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, is written as $\mathbb{Z}/2\mathbb{Z}^2$ or $\mathbb{Z}_2^2$ for shorthand notation. It consists of ordered pairs of 0's and 1's with component-wise addition mod 2.

Similar to the order of permutations in cycle notations, if for $a \in G$, $b \in H$ we have $o(a) = m$ and $o(b) = n$ then the order of (a,b) in $G \oplus H$ is $\operatorname{lcm}(m,n)$.

Proposition 0.0.2

If $G = \langle a \rangle$ and $H = \langle b \rangle$ then $G \oplus H$ is cyclic if and only if $\langle G \rangle, \langle h \rangle$ are co-prime. *Clear enough.*

Example 0.0.3. Let m and n be co-prime. Let $f : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ be defined by $x \mapsto (x + m\mathbb{Z}, x + n\mathbb{Z})$. Then f is a homomorphism. Note that $\ker(f) = mn\mathbb{Z}$.
Then, the F.I.T. gives us the following:

$$\mathbb{Z}/mn\mathbb{Z} = f(\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}.$$

Example 0.0.4. $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \ncong \mathbb{Z}/32\mathbb{Z}$ as the former has <u>no</u> element of order 32.

Example 0.0.5. Some of the groups of order 8 include:

$$D_4, \quad \mathbb{Z}/8\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z}^3.$$

There is one more called the *quarternion group*.

Most of the times, set-theoretic constructions do <u>not</u> create a group. In general, even if H, K are subgroups of G,

$$HK = \{hk \mid h \in H, k \in K\}$$

is not a group. However, normality can change things:

Lemma 0.0.5.1

If H, K are subgroups of G and $H \lhd G$ then HK <u>is</u> a subgroup. One between two being normal suffices.

Proof. Claim: if $H \lhd G$ then $HK = KH$. Indeed, for all $k \in K$ we have $Hk = kH$.

(1) Identity: trivial.

(2) Inverse: $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$.

(3) Closure: $(hk)(h'k') = h(kh')k' = h(\tilde{h}'\tilde{k})k' = (h\tilde{h}')(\tilde{k}k') \in HK$.

□