MATH 425b Problem Set 1

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Problem 1

(a) Prove that limits of nets in $\mathbb{R}$ are unique: if $f : A \to \mathbb{R}$ is a net, $\lim f = L$, and $\lim f = L'$, then $L = L'$.

Proof. Given $\epsilon > 0$, there exist $a_1, a_2 \in A$ such that

$$\text{for all } a \in A, \begin{cases} a_1 \leq a \implies |f(a) - L| < \epsilon/2 \\ a_2 \leq a \implies |f(a) - L'| < \epsilon/2 \end{cases}.$$

By triangle inequality, if $a_1 \leq \tilde{a}$ and $a_2 \leq \tilde{a}$ (which is always possible since upper bounds exist; in fact defining $\tilde{a} := a_2$ if $a_1 \leq a_2$ and $\tilde{a} := a_1$ otherwise suffice), $|L - L'| \leqslant |L - f(\tilde{a})| + |f(\tilde{a}) - L'| < \epsilon$. Since ϵ is arbitrary, $L = L'$. □

(b) Let $f, g : A \to \mathbb{R}$ be nets in $\mathbb{R}$. Prove that if $\lim f$ and $\lim g$ exist and $c \in \mathbb{R}$, then

$$\lim(f + cg) = \lim f + c \lim g, \ \ \lim(f \cdot g) = \lim f \cdot \lim g, \ \text{ and } \ \lim(f/g) = (\lim f)/(\lim g)$$

where, for the last one, g is assumed to be nonvanishing with $\lim g \neq 0$.

Proof. Define $F := \lim f$ and $G := \lim g$.

(I) Given $\epsilon > 0$, there exist $a_1, a_2 \in A$ such that

$$\text{for all } a \in A \begin{cases} a_1 \leq a \implies |f(a) - F| < \epsilon/(c+1) \\ a_2 \leq a \implies |g(a) - G| < \epsilon/(c+1) \end{cases}.$$

Then for any $\tilde{a} \in A$ such that $a_1 \leq a \wedge a_2 \leq a$, we have

$$\begin{aligned} |(f + cg)(\tilde{a}) - (\lim f + c \lim g)| &= |f(\tilde{a}) + cg(\tilde{a}) - F - cG| \\ &\leqslant |f(\tilde{a}) - F| + |cg(\tilde{a}) - cG| \\ &< \frac{\epsilon}{c+1} + \frac{c\epsilon}{c+1} = \epsilon. \end{aligned}$$

Since ϵ is arbitrary, $\lim(f + cg) = F + cG$, as desired.

(II) (Reverse thinking here; I dislike proofs that present a complicated δ like "magic" before actually deriving the $<\epsilon$ inequality. Therefore I would like to go "backwards" and show *where* that complicated δ comes from.) By using triangle inequality multiple times, we have

$$\begin{aligned}|f(a)g(a)-FG| &= |f(a)g(a)-f(a)G+f(a)G-FG| \\ &\leqslant |f(a)g(a)-f(a)G|+|f(a)G-FG| \\ &\leqslant |(f(a)-F)(g(a)-G)|+|Fg(a)-FG|+|f(a)G-FG| \\ &\leqslant \underbrace{|f(a)-F||g(a)-G|}_{\text{both can be bounded}}+\underbrace{|F|}_{<\infty}\underbrace{|g(a)-G|}_{\text{bounded}}+\underbrace{|G|}_{<\infty}\underbrace{|f(a)-F|}_{\text{bounded}}.\end{aligned}$$

From this we already see that $(f\cdot g)(a)$ can be arbitrarily close to FG, i.e., $\lim(f\cdot g)=\lim f\cdot\lim g$. Explicit solution: given $\epsilon>0$, $|f(a)g(a)-FG|<\epsilon$ whenever $|f(a)-F|<\delta$ and $|g(a)-G|<\delta$ where

$$\delta := \frac{\min(1,\epsilon)}{1+|F|+|G|}.$$

(III) Let $\epsilon>0$ be given. We have

$$\left|\frac{1}{g(x)}-\frac{1}{G}\right| = \left|\frac{G-g(x)}{Gg(x)}\right| = \underbrace{\frac{1}{|g(x)|}}_{\text{need to bound}}\overbrace{\frac{1}{|G|}|G-g(x)|}^{\text{one finite, one bounded}}.$$

Since $\lim g=G$, there exist $a_1,a_2\in A$ such that

$$\text{for all } a\in A \begin{cases} a_1\preceq a \implies |g(a)-G|<|G|/2 \\ a_2\preceq a \implies |g(a)-G|<|G|^2\epsilon/2 \end{cases}.$$

Let $\tilde{a}\in A$ be an upper bound of a_1 and a_2. It follows that $|g(a)-G|<\min(|G|/2,|G|^2\epsilon/2)$. Therefore,

$$|G| \overset{\Delta\text{-ineq}}{\leqslant} \underbrace{|G-g(\tilde{a})|}_{<|G|/2}+|g(\tilde{a})| \implies |g(\tilde{a})|>|G|/2 \implies \frac{1}{|g(\tilde{a})|}<\frac{2}{|G|},$$

and thus

$$\left|\frac{1}{g(\tilde{a})}-\frac{1}{G}\right| < \frac{2}{|G|}\frac{1}{|G|}\frac{|G^2|\epsilon}{2}=\epsilon.$$

Therefore we've just shown $\lim(1/g)=1/\lim g$, and the claim of (III) follows from applying (II). □

(c) For a closed interval $[a,b]$, let A be the set of partition pairs (P,T) of $[a,b]$. Define $(P,T)\preceq(P',T')$ when P' is a definement of P. Show that A is a directed set.

Proof. Criterion (1) is trivially true $P \subseteq P$. Criterion (2) is also true since $P \subset P'$ and $P' \subset P''$ implies $P \subset P''$. For criterion (3), suppose $(P,T) \preceq (P_1,T_1)$ and $(P,T) \preceq (P_2,T_2)$. Clearly if we let $P_3 := P_1 \cup P_2$ and define T_3 accordingly (i.e., depending on P_3), then $P \subset P_1 \subset P_3$ and $P \subset P_2 \subset P_3$ imply $(P,T) \preceq (P_3,T_3)$. □

(d) For a function $f : [a,b] \to \mathbb{R}$, the assignment $(P,T) \mapsto R(f,P,T)$ is a net from A to $\mathbb{R}$. Prove that f is Riemann integrable with integral I if and only if this net converges to I.

Proof. $\Longrightarrow$: suppose that f is Riemann integrable and let $\epsilon > 0$ be given. Then there exists some $\delta > 0$ such that whenever $\operatorname{mesh}(P) < \delta$, $|R(f,P,T),I| < \epsilon$. Let (P_0,T_0) be one of these partition pairs with $\operatorname{mesh}(P) < \delta$. Suppose that the net does *not* converge to I. Then there exists $\epsilon > 0$ such that, for all partition pairs, in particular (P_0,T_0), there exists $(\tilde{P},\tilde{T})$ with $(P_0,T_0) \preceq (\tilde{P},\tilde{T})$, i.e., $P_0 \subset \tilde{P}$, such that $|R(f,\tilde{P},\tilde{T}) - I| \geqslant \epsilon$. Notice that $P_0 \subset \tilde{P}$ means exactly $\operatorname{mesh}(\tilde{P}) \leqslant \operatorname{mesh}(P_0) < \delta$. Hence we simutaneously have $|R(f,\tilde{P},\tilde{T}) - I| < \epsilon$ and $|R(f,\tilde{P},\tilde{T}) - I| \geqslant \epsilon$, clearly a contradiction, so the net must converge to I.

$\Longleftarrow$: suppose the net converges to I. Let $\epsilon > 0$ be given. By assumption we can find $(P_0,T_0) := a \in A$ such that $|R(f,\tilde{P},\tilde{T}) - I| < \epsilon/2$ whenever $P_0 \subset \tilde{P}$. Notice that the net has no additional requirements on $\tilde{T}$ as long as it forms a partition pair with $\tilde{P}$. Therefore, for such $\tilde{P}$ we have

$$\begin{cases} |U(f,\tilde{P}) - I| < \epsilon/2 \\ |L(f,\tilde{P}) - I| < \epsilon/2 \end{cases} \xRightarrow{\Delta\text{-ineq}} |U(f,\tilde{P}) - L(f,\tilde{P})| < \epsilon.$$

Therefore f is Riemann integrable by *Riemann's Integrability Criterion.* □

(e) Prove that if $f \in \mathcal{R}[a,b]$ then its Riemann integral I is unique.

Proof. Suppose f has Riemann integrals I and I'. Then

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}x = I \wedge \int_a^b f(x)\,\mathrm{d}x = I' &\overset{\text{(d)}}{\iff} \text{the corresponding nets converge to } I \,\&\, I' \\ &\overset{\text{(a)}}{\iff} I = I'. \end{aligned}$$ □

(f) Prove that if $f,g \in \mathcal{R}[a,b]$ with Riemann integrals I and J respectively and $c \in \mathbb{R}$, then $f + cg \in \mathcal{R}[a,b]$ with Riemann integral $I + cJ$.

Proof. Define $\overline{f}$ by $(P,T) \mapsto R(f,P,T)$ and $\overline{g}$ by $(P,T) \mapsto R(g,P,T)$. Using (d) and (I) from (b), we have

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}x = I \wedge \int_a^b g(x)\,\mathrm{d}x = J &\overset{\text{(d)}}{\iff} \lim \overline{f} = I \wedge \lim \overline{g} = J \\ &\overset{\text{(b).(I)}}{\Longrightarrow} \lim(\overline{f} + c\overline{g}) = I + cJ \\ &\overset{\text{(d)}}{\iff} \int_a^b f(x) + cg(x)\,\mathrm{d}x = I + cJ \end{aligned}$$

(and of course $f + cg \in \mathcal{R}[a,b]$ as implied by (d) as well). □

(g) Generalize the definitions of nets and convergence of nets from the case of $\mathbb{R}$ to the case of a general metric space X. (One can generalize even further to topological spaces X; if X is a Hausdorff space, then limits of nets in X are unique.)

Solution: generalized defintion. Let (X, d) be a metric space. We say $f : A \to X$ *converges to a limit* L, denoted as $\lim f = L$ if, for every $\epsilon > 0$, there exists $a_0 \in A$ such that for all $a \in A$ with $a_0 \preceq a$, we have $d(f(a), L) < \epsilon$.

For convergence in Hausdorff spaces, we need to first generalize the definition again: for any neighborhood U of L, there exists $a_0 \in A$ such that $a \in U$ whenever $a_0 \preceq a$.

Proof: Hausdorff $\implies$ unique limits. Suppose for contradiction that f converges to distinct L and L'. On one hand, since H is Hausdorff, there exist neighborhoods U of L and U' of L' such that $U \cap U' = \varnothing$. On the other hand, by convergence of f, there exists $a_1, a_2 \in A$ such that

$$\text{for all } a \in A, \begin{cases} a_1 \preceq a \implies a \in U \\ a_2 \preceq a \implies a \in U' \end{cases}.$$

If we take any upper bound of a_1 and a_2, say $\tilde{a}$, we immediately get a contradiction that $a \in U \cap U' = \varnothing$. Therefore $L = L'$, and limits of nets in Hausdorff spaces are unique. □

Problem 2

Prove that if $f, g \in \mathcal{R}$ and $f \leqslant g$ then

$$\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x.$$

Proof. By the hint we first consider the convergent (since f, g are R.I.) nets $f, g : A \to \mathbb{R}$. By linearity ((b,II) & (b,III) above), the difference $g - f : A \to \mathbb{R}$ is also convergent with $\lim(g - f) = \lim g - \lim f$. Since $f(a) \leqslant g(a)$ for all $a \in A$, $(g - f)(a) \geqslant 0$ for all $a \in A$. It follows that $\lim(g - f)$ *cannot* be negative.

Suppose $\lim(g - f) = -M < 0$, then taking $\epsilon := M/2$ gives a contradiction of the convergence of $g - f$, for $|(g - f)(a) - (-M)| < \epsilon \iff -1.5M < (g - f)(a) < -0.5M < 0$ whereas $(g - f)$ is nonnegative, meaning no $a \in A$ satisfies this inequality.

Therefore $\lim g - \lim f \geqslant 0 \implies \lim f \leqslant \lim g$, and by (d) we conclude that $\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x$. □