

## HOMework, WEEK 1

This assignment is due Friday, January 22 in lecture. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine). Note that an equation such as  $2 + 2 = 4$  can be read as the statement “two plus two equals four.” Chains of equations such as

$$\begin{aligned}(1 + 2) + (3 + 4) &= 3 + 7 \\ &= 10 \\ &= 2 \cdot 5\end{aligned}$$

can be read as “(one plus two) plus (three plus four) equals three plus seven, which equals ten, which equals two times five.” The important thing is that equations or chains of equations need to be properly integrated into a correct and complete argument with logical connecting words or logical symbols, rather than just stated on their own.

The ideal solutions are as short and clear as possible while containing all the required ingredients and logical connectives to form a valid argument. After this assignment is graded, some of you might find that you need to include more detail in your proofs. Others might find that your proofs could be made more concise by omitting trivial or irrelevant details. Ideally, you will develop your proof-writing skills over time, building on what you’ve learned in 425a and other proof-based courses.

- (1) This problem will allow you to rigorously make sense of Riemann integrals as “limits,” letting you apply general facts about limits to prove properties of Riemann integrals. One setup that encompasses the Riemann integral as well as other limits you have seen is the concept of nets (a generalization of sequences) and their limits.

Let  $A$  be a set with a binary relation  $\preceq$  satisfying

- $a \preceq a$  for all  $a \in A$  (reflexivity),
- if  $a \preceq b$  and  $b \preceq c$ , then  $a \preceq c$  (transitivity), and
- for any  $a, b \in A$  there exists  $c \in A$  with  $a \preceq c$  and  $b \preceq c$  (upper bounds exist)

(such  $A$  is called a *directed set*). A function  $f : A \rightarrow \mathbb{R}$  is called a *net* from  $A$  to  $\mathbb{R}$ .

If  $f$  is a net from  $A$  to  $\mathbb{R}$ , we say that  $f$  *converges to a limit*  $L$  if for every  $\varepsilon > 0$ , there exists an element  $a_0 \in A$  such that for all  $a \in A$  with  $a_0 \preceq a$ , we have  $|f(a) - L| < \varepsilon$ . We write  $\lim f = L$ .

For some examples:

- For  $A = \mathbb{N}$  with its usual order relation, a net from  $A$  to  $\mathbb{R}$  is a sequence in  $\mathbb{R}$ , and convergence of the net is equivalent to convergence of the sequence.
- For  $A = \mathbb{R}$  with its usual order relation, a net from  $A$  to  $\mathbb{R}$  is a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , and convergence of the net is equivalence to existence of the limit  $\lim_{t \rightarrow \infty} f(t)$ .
- Fix a point  $x \in \mathbb{R}$ , and define a relation  $\preceq$  on  $A = \mathbb{R} \setminus \{x\}$  by  $t_1 \preceq t_2$  when  $|t_2 - x| \leq |t_1 - x|$ . A net from  $A$  to  $\mathbb{R}$  is a function  $f : \mathbb{R} \setminus \{x\} \rightarrow \mathbb{R}$ , and convergence of the net is equivalent to existence of the limit  $\lim_{t \rightarrow x} f(t)$ .

- (a) Prove that limits of nets in  $\mathbb{R}$  are unique: if  $f : A \rightarrow \mathbb{R}$  is a net,  $\lim f = L$ , and  $\lim f = L'$ , then  $L = L'$ .
- (b) Let  $f, g : A \rightarrow \mathbb{R}$  be nets in  $\mathbb{R}$ . Prove that if  $\lim f$  and  $\lim g$  exist and  $c$  is a real number, then we have:
- $\lim(f + cg) = \lim f + c \lim g$
  - $\lim(f \cdot g) = \lim f \cdot \lim g$
  - $\lim(f/g) = (\lim f)/(\lim g)$ ,
- where in the last equation  $g$  is assumed to be a nonvanishing function with  $\lim g$  also nonzero.
- (c) For a closed interval  $[a, b]$ , let  $A$  be the set of partition pairs / tagged partitions  $(P, T)$  of  $[a, b]$ . Define  $(P, T) \preceq (P', T')$  when  $P'$  is a refinement of  $P$ . Show that  $A$  is a directed set.
- (d) For a function  $f : [a, b] \rightarrow \mathbb{R}$ , the assignment  $(P, T) \mapsto R(f, P, T)$  is a net from  $A$  to  $\mathbb{R}$ . Prove that  $f$  is Riemann integrable with integral  $I$  if and only if this net converges to  $I$ . (*Hint*: the harder direction is (convergence of the net)  $\Rightarrow$  Riemann integrability. Luckily, you can make use of Riemann's integrability criterion (Chapter 3, Theorem 21 in Pugh) without circularity; this will be easier than using the definition of Riemann integrability directly.)
- (e) Prove that if  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable, then its Riemann integral  $I$  is unique. (*Hint*: use earlier parts of the problem to make this easy, although you could also prove it directly.)
- (f) Prove that if  $f : [a, b] \rightarrow \mathbb{R}$  and  $g : [a, b] \rightarrow \mathbb{R}$  are Riemann integrable with Riemann integrals  $I$  and  $J$  respectively, and  $c \in \mathbb{R}$ , then  $f + cg$  is Riemann integrable with Riemann integral  $I + cJ$ . (*Hint*: again, use earlier parts and the relation  $R(f + cg, P, T) = R(f, P, T) + cR(g, P, T)$ .)
- (g) Generalize the definitions of nets and convergence of nets from the case of  $\mathbb{R}$  to the case of a general metric space  $X$ . (One can generalize even further to topological spaces  $X$ ; if  $X$  is a Hausdorff space, then limits of nets in  $X$  are unique. You don't need to prove anything about your generalized definition here.)
- (2) Prove Theorem 18 in Chapter 3 of Pugh (monotonicity of the Riemann integral). Pugh gives a sketch of a proof; you should fill it in with details. You have a choice of how to proceed; you can either work directly from the definition of Riemann integrability, or you can prove that if  $f, g : A \rightarrow \mathbb{R}$  are convergent nets in  $\mathbb{R}$  with  $f(a) \leq g(a)$  for all  $a \in A$ , then

$$\lim f \leq \lim g,$$

and use this general result to prove monotonicity for Riemann integrals.