Problem 0.0.1

Prove that all norms on a finite-dimensional vector space are equivalent.

Proof. Let V be a vector space with $\beta = \{e_1, \dots, e_n\}$ a basis for it. Define

$$\|x\|_\beta = \left(\sum_{i=1}^n |\alpha_i|^2\right)^{1/2} \quad \text{where } x = \sum_{i=1}^n \alpha_i e_i.$$

It immediately follows that $\|\cdot\|_\beta$ defines a norm on V. Furthermore, $(V, \|\cdot\|_\beta) \equiv \mathbb{R}^n$ (isometrically isomorphic). To see this, consider $T : \mathbb{R}^n \to V$ defined by

$$T(\alpha_1, \dots, \alpha_n) := \sum_{i=1}^n \alpha_i e_i.$$

Since β is a basis of V, each $x \in V$ is uniquely represented by this basis. Hence T is linear and bijective. In addition,

$$\|T(\alpha_1, \dots, \alpha_n)\|_\beta = \left(\sum_{i=1}^n |\alpha_i|^2\right)^{1/2} = |(\alpha_1, \dots, \alpha_n)|_{\text{st}}.$$

(The RHS is the standard Euclidean norm on $\mathbb{R}^n$.) Therefore we have used β to identify V with $\mathbb{R}^n$.

Now, let $\|\cdot\|$ be any other norm. By Cauchy Schwarz, we have

$$\|x\| = \left\| \sum_{i=1}^n \alpha_i e_i \right\| \leqslant \sum_{i=1}^n |\alpha_i| \|e_i\| \leqslant \underbrace{\left(\sum_{i=1}^n |\alpha_i|^2\right)^{\frac{1}{2}}}_{\|x\|_\beta} \underbrace{\left(\sum_{i=1}^n \|e_i\|^2\right)^{\frac{1}{2}}}_{:=c_2 \text{ a constant}} = c_2 \|x\|_\beta. \tag{1}$$

Now it remains to show that there exists a $c_1 > 0$ such that $c_1 \|x\|_\beta \leqslant \|x\|$. We first show that the mapping $x \mapsto \|x\|$ is continuous with respect to $\|\cdot\|_\beta$: indeed, by triangle inequality we have

$$\begin{cases} \|x\| \leqslant \|x - y\| + \|y\| \\ \|y\| \leqslant \|x\| + \|y - x\| \end{cases} \implies \begin{cases} \|x\| - \|y\| \leqslant \|x - y\| \\ \|y\| - \|x\| \leqslant \|y - x\| \end{cases} \implies |\|x\| - \|y\|| \leqslant \|x - y\| \leqslant c_2 \|x - y\|_\beta.$$

Now consider the unit sphere with respect to $\|\cdot\|_\beta$:

$$\mathcal{S} := \{v \in V : \|v\|_\beta = 1\}.$$

Certainly, by the isometry to $\mathbb{R}^n$ and Heine-Borel theorem, $\mathcal{S}$ is compact. Therefore $\|x\|$ is bounded on $\mathcal{S}$ and it also attain its bounds. In particular, the lower bound $c_1 > 0$ because it is the norm of some $x' \in \mathcal{S}$ (so $\|x'\| \neq 0$ or otherwise $x' = 0 \notin \mathcal{S}$). For any arbitrary (nonzero) $x \in V$, we have

$$c_1 \leqslant \left\| \frac{x}{\|x\|_\beta} \right\| = \frac{\|x\|}{\|x\|_\beta} \implies c_1 \|x\|_\beta \leqslant \|x\|. \tag{2}$$

Combining (1) and (2) we see that any arbitrary $\|\cdot\|$ is equivalent to $\|\cdot\|_\beta$. Now, let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on V. We know both are equivalent to $\|\cdot\|_\beta$. Therefore there exist $c_1, c_2, d_1, d_2 > 0$ such that

$$\begin{cases} c_1 \|x\|_\beta \leqslant \|x\|_1 \leqslant c_2 \|x\|_\beta \\ d_1 \|x\|_\beta \leqslant \|x\|_2 \leqslant d_2 \|x\|_\beta \end{cases} \implies \frac{c_1}{d_2} \|x\|_2 \leqslant \|x\|_1 \leqslant \frac{c_2}{d_1} \|x\|_2.$$

Therefore $\|\cdot\|_1, \|\cdot\|_2$ are equivalent, and the claim follows. $\square$

Problem 0.0.2

Show heuristically that differentiating a function f corresponds to multiplying its Fourier transform $\hat{f}(\xi)$ by the linear function $2\pi i\xi$, i.e., that

$$\widehat{f'} = 2\pi i\xi \hat{f}(\xi).$$

Proof. With the assumptions provided by the hint,

$$\begin{aligned}\widehat{f'}(\xi) &= \int_{-\infty}^{\infty} f'(\tilde{x})e^{-2\pi i\xi\tilde{x}}\,\mathrm{d}\tilde{x} \\ &= f(\tilde{x})e^{-2\pi i\xi\tilde{x}}\Big|_{\tilde{x}=-\infty}^{\infty} - (-2\pi i\xi)\int_{-\infty}^{\infty} f(\tilde{x})e^{-2\pi i\xi\tilde{x}}\,\mathrm{d}\tilde{x} \\ &= 2\pi i\xi\hat{f}(\xi).\end{aligned} \qquad \begin{bmatrix} u = e^{-2\pi i\xi x} & \mathrm{d}u = -2\pi i\xi e^{-2\pi i\xi x}\mathrm{d}x \\ \mathrm{d}v = f'(x)\mathrm{d}x & v = f(x)\end{bmatrix}$$

□

Problem 0.0.3

Show heuristically that, for sufficiently nice $f : \mathbb{R} \to \mathbb{C}$, if

$$g := \frac{\mathrm{d}^n f}{\mathrm{d}x^n} f + c_{n-1}\frac{\mathrm{d}^{n-1} f}{\mathrm{d}x^{n-1}} + \cdots + c_0 f,$$

then

$$\hat{g}(\xi) = \left[(2\pi i\xi)^n + c_{n-1}(2\pi i\xi)^{n-1} + \cdots + c_0\right]\hat{f}(\xi).$$

Proof. By applying the result from the previous problem inductively, we have

$$\widehat{f^{(k)}}(\xi) = (2\pi i\xi)^k \hat{f}(\xi).$$

It is also clear that $\widehat{(cf)}(\xi) = c\hat{f}(\xi)$. The claim the follows. □