# HOMEWORK, WEEK 10

This assignment is due Monday, March 29. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

(1) Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ and let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on $V$. Prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ are comparable, i.e. that there exist $C_1, C_2 > 0$ such that $\|v\|_1 \leq C_1\|v\|_2$ and $\|v\|_2 \leq C_2\|v\|_1$ for all $v \in V$.

*Hint:* It's best to approach this problem in multiple steps. Let $\beta = \{e_1, \ldots, e_n\}$ be a basis for $V$; define a norm $\|\cdot\|_\beta$ on $V$ by $\|\sum_{i=1}^n v^i e_i\|_\beta = (\sum_{i=1}^n (v^i)^2)^{1/2}$ (in other words, you use $\beta$ to identify $V$ with $\mathbb{R}^n$, and $\|\cdot\|_\beta$ comes from the standard Euclidean norm on $\mathbb{R}^n$).

The first step is to show that the function $\|\cdot\|_1 : V \to \mathbb{R}$ is continuous when $V$ is given the metric induced by $\|\cdot\|_\beta$. Indeed, you can even show it's uniformly continuous. Given $\varepsilon > 0$, you want to find $\delta > 0$ such that if $v, w \in V$ and $\|v-w\|_\beta < \delta$, then $|\|v\|_1 - \|w\|_1| < \varepsilon$. Show that $|\|v\|_1 - \|w\|_1| \leq \|v - w\|_1$ follows from the triangle inequality for $\|\cdot\|_1$ (rewrite it as a double inequality).

Since $\|\cdot\|_\beta$ is defined in terms of basis coordinates, expand out $v$ and $w$ in coordinates and use the triangle inequality for $\|\cdot\|_1$ to estimate $\|v - w\|_1$. Try to find $\delta$ such that if the Euclidean distance between (the coordinate vectors of) $v$ and $w$ is less than $\delta$, then $\|v - w\|_1 < \varepsilon$.

Thus, $\|\cdot\|_1$ is continuous from $(V, \|\cdot\|_\beta)$ to $\mathbb{R}$. Consider the restriction of $\|\cdot\|_1$ to the $\|\cdot\|_\beta$-unit sphere $S = \{v \in V \mid \|v\|_\beta = 1\}$. Use the Heine–Borel theorem to show that $S$ is compact. From there, show that there exist $m, M > 0$ such that for all $v \in S$, we have $m \leq \|v\|_1 \leq M$.

Now you can show that $\|\cdot\|_1$ and $\|\cdot\|_\beta$ are comparable. Indeed, show that for all $v \in V$, we have $\|v\|_1 \leq M\|v\|_\beta$ and $\|v\|_\beta \leq (1/m)\|v\|_1$. When $v = 0$, the inequalities are equalities; assume $v \neq 0$ and consider $u = \frac{v}{\|v\|_\beta} \in S$. Apply the previous paragraph to $u$.

It follows that $\|\cdot\|_\beta$ and $\|\cdot\|_1$ are comparable; equally well, $\|\cdot\|_\beta$ and $\|\cdot\|_2$ are comparable. To finish the problem, show that comparability of norms is a transitive relation (if $\|\cdot\|_1$ is comparable to $\|\cdot\|_2$ and $\|\cdot\|_2$ is comparable to $\|\cdot\|_3$, then $\|\cdot\|_1$ is comparable to $\|\cdot\|_3$). Since comparability is clearly reflexive and symmetric, comparability is an equivalence relation on norms. Conclude that the given norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $V$ are comparable.

(2) We return to Fourier analysis for another round of non-rigorous computations. The Fourier transform is very important to know about, although it takes work to set up precisely (while the usual treatments are based on Lebesgue integration, Stein and Shakarchi give a treatment that avoids it). Along with Fourier transforms, one must

often consider Dirac $\delta$-functions which are treated using Laurent Schwartz's theory of distributions, again requiring some technical machinery.

In this problem we will consider Fourier transforms heuristically for functions $f : \mathbb{R} \to \mathbb{C}$. In the general theory of Pontryagin duality, $\widehat{\mathbb{R}} \cong \mathbb{R}$, so $f$ should be "dual" to another function $\hat{f} : \mathbb{R} \to \mathbb{C}$ (the Fourier transform of $f$). The general theory suggests the following definition: for $\xi \in \mathbb{R}$, define

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

(we will not worry for now about whether such integrals make sense). The values $\hat{f}(\xi)$ of the Fourier transform of $f$ are analogous to the Fourier coefficients of a periodic function, but they are defined for all real $\xi$ rather than just for integers $k$, and the integral is over the whole real line.

For nice enough periodic functions, we could write $f$ as the limit of a (convergent) Fourier series; analogously, for nice enough $f : \mathbb{R} \to \mathbb{C}$, we have the "Fourier inversion formula"

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

In particular, for $f = \delta_0$, we have $\hat{\delta}_0(\xi) = e^{-2\pi i \xi \cdot 0} = 1$ for all $\xi$, and the Fourier inversion formula reads

$$\delta_0(x) = \int_{-\infty}^{\infty} e^{2\pi i \xi x} d\xi.$$

This special case of the formula implies the general case (heuristically), since

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(y) \delta_0(x - y) dy \\ &= \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{2\pi i \xi (x-y)} d\xi dy \\ &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(y) e^{-2\pi i \xi y} dy \right) e^{2\pi i \xi x} d\xi \\ &= \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi. \end{aligned}$$

Convolution of functions $f, g : \mathbb{R} \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x - y) dy,$$

and it corresponds to multiplication of Fourier transforms: we have

$$\widehat{f * g}(\xi) = \hat{f}(\xi) \hat{g}(\xi)$$

(note that $f * g = g * f$).

**Problem.** Show (heuristically) that differentiating a function $f$ corresponds to multiplying its Fourier transform $\hat{f}(\xi)$ by the linear function $2\pi i\xi$, i.e. that

$$\widehat{\frac{df}{dx}}(\xi) = 2\pi i\xi \hat{f}(\xi).$$

You may use integration by parts while taking all boundary values at $\pm\infty$ to be zero, as well as other heuristic manipulations like differentiating under integral signs.

*Hint:* Once you move the derivative $\frac{d}{dx}$ from $f$ to the exponential factor in the integral defining $\widehat{\frac{df}{dx}}(\xi)$, you know how to differentiate exponential functions.

(3) The result of the above problem is very useful when solving differential equations, especially in the case of equations like

$$\frac{d^n f}{dx^n} + c_{n-1}\frac{d^{n-1}f}{dx^{n-1}} + \cdots + c_0 f = g,$$

where $f$ is the unknown function and $g$ is some given function (the "inhomogeneous term" of a linear inhomogeneous differential equation).

**Problem.** Show heuristically that if $f$ is a (sufficiently nice) function from $\mathbb{R}$ to $\mathbb{C}$ with Fourier transform $\hat{f}$, then

$$\left(\frac{d^n f}{dx^n} + c_{n-1}\frac{d^{n-1}f}{dx^{n-1}} + \cdots + c_0 f\right)^{\wedge}(\xi) = \left((2\pi i\xi)^n + c_{n-1}(2\pi i\xi)^{n-1} + \cdots + c_0\right)\hat{f}(\xi).$$

*Hint:* Just apply the previous problem.

(4) If we can find all solutions to the homogeneous equation

$$\frac{d^n f}{dx^n} + c_{n-1}\frac{d^{n-1}f}{dx^{n-1}} + \cdots + c_0 f = 0$$

(e.g. by finding $n$ independent solutions, at which point the Picard–Lindelöf theorem guarantees we've found them all), then to find all solutions to the original inhomogeneous equation, we only need to find one solution (all others are obtained by adding a solution of the homogeneous equation).

If $f$ solves the inhomogeneous equation for a given inhomogeneous term $g$, the previous problem implies that we can write

$$\left((2\pi i\xi)^n + c_{n-1}(2\pi i\xi)^{n-1} + \cdots + c_0\right)\hat{f}(\xi) = \hat{g}(\xi).$$

If we are looking for a solution to the equation with inhomogeneous term $g$, we can use the above equation to solve for $\hat{f}(\xi)$ and then recover $f$ using Fourier inversion.

**Problem.** Using Fourier analysis (rather than another technique like undetermined coefficients or variation of parameters), find all solutions $f : \mathbb{R} \to \mathbb{C}$ to the differential equation

$$f''(t) + 4f'(t) + 3f(t) = e^{4\pi i t}.$$

(i.e. all differentiable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying the equation). Prove rigorously that your answer is correct. You may use the Picard–Lindelof theorem with global uniqueness for vector fields $F : U \to \mathbb{C}^m$ where $U \subset \mathbb{C}^m$ is open (this version follows from the real version and the identification $\mathbb{C}^m \cong \mathbb{R}^{2m}$). When arguing about uniqueness of solutions to $n^{th}$-order differential equations, you do not need to go into detail about how to translate between higher-order equations and the corresponding first-order equations.

*Hint:* Use heuristic reasoning to find the solutions first, then do the rigorous part later. Compute the Fourier transform $\hat{g}(\xi)$ where $g(t) = e^{4\pi it}$; it should be a delta function. Then, using the previous problem and dividing, you can get a formula for $\hat{f}(\xi)$. Plug this into the Fourier inversion formula to get a formula for $f(t)$; the presence of the $\delta$ function should get rid of the integral without the need for any computations.

Once you've found your solution $f$, check rigorously that it's actually a solution. Now show (as mentioned above) that any other solution $\tilde{f}$ must be equal to $f + h$ for some $h$ solving the homogeneous equation $D(h) = 0$. Find two linearly independent solutions to the homogeneous equation $f'' + 4f' + 3f = 0$; check rigorously that they're solutions, and show that their initial-condition vectors $(f(0), f'(0))$ are linearly independent in $\mathbb{R}^2$. Use Picard's theorem to show that any solution $h$ to the homogeneous equation must be a linear combination of your two explicit solutions.

**Remark.** Of course, the above analysis assumes that $\hat{g}(\xi)$ makes sense for $\xi$ on the real line—this is more plausible for some inhomogeneous terms $g$ than for others. The Fourier transform as we've formulated is especially useful for periodic or "steady-state" inhomogeneous terms like $e^{4\pi it}$, since the Fourier transform for such a function makes sense and is a Dirac delta function on the real line. These differential equations commonly arise as the equation satisfied by the output voltage of some circuit whose input voltage is $g(t)$, and Fourier analysis is useful for understanding the steady-state frequency response of the circuit.

When the inhomogeneous term is of a more general form, say $e^{4t}$ rather than $e^{4\pi it}$, then existence of the Fourier transform is more subtle (e.g. for $g(t) = e^{4t}$, we should really be looking at $\hat{g}(\xi)$ for $\xi$ of the form $a - \frac{2}{\pi}i$ for $a \in \mathbb{R}$). Since we are allowing complex $\xi$, we can equivalently view things in terms of the "bilateral Laplace transform"

$$\mathcal{B}(g)(s) := \int_{-\infty}^{\infty} g(t)e^{-st}dt$$

which is typically only defined on a subset (e.g. a line) of $s$-values in the complex plane ($s$ and $\xi$ are related by $s = 2\pi i\xi$). A more general version of Fourier inversion, the Mellin inversion formula, then lets us reconstruct a function from its bilateral Laplace transform.

An especially important type of inhomogeneous term in applications is a function $g(t)$ which is zero for $t < 0$ but becomes nonzero for $t \geq 0$ (think of a circuit sitting on the table idle until you push the "on" button). For such functions, the integral from $-\infty$ to 0 in the bilateral Laplace transform $\mathcal{B}(g)(s)$ vanishes, so we get the ordinary Laplace transform

$$\mathcal{L}(g)(s) := \int_{0}^{\infty} g(t)e^{-st}dt.$$

For general $g$, $\mathcal{L}(g)$ is often defined on a larger subset of the complex plane than is $\mathcal{B}(g)$ (e.g. a half-plane rather than just a line).

A useful fact to remember is that if $g(t)$ grows like $e^{ct}$ for $c \in \mathbb{R}$, then $\mathcal{L}(g)(s)$ is defined on the half-plane consisting of $s$ with real part $\geq c$. For e.g. the bilateral Laplace transform applied to $g(t) = e^{ct}$, we need both integrals $\int_{-\infty}^{0}$ and $\int_{0}^{\infty}$ to exist, so we should restrict to the intersection of two half planes, namely the line of $s$ with real part exactly equal to $c$. Another related fact is that $g$ is compactly supported, i.e. $g$ vanishes for both sufficiently large negative $t$ and sufficiently large positive $t$, then the bilateral Laplace transform of $g$ exists (and is holomorphic) on the whole complex plane (equivalently, this is true for the Fourier transform of $g$; modulo details this is called the Paley–Weiner theorem).

Even for homogeneous terms $g(t)$ which do not vanish for $t < 0$, it is common to solve inhomogeneous ODEs using the Laplace transform rather than Fourier transforms. The reasoning is similar to what we did in this problem, although it's not entirely parallel; due to the boundary term at zero, Laplace transforms turn derivatives into slightly more complicated polynomial expressions, and to perform the inversion step one typically uses a partial-fractions decomposition plus a knowledge of how to invert the pieces that arise.

(5) An interesting fact conceptually is that to consider different inhomogeneous terms $g$ separately: we can focus on the special case where $g$ is a Dirac delta function and deduce everything else from that case.

**Definition.** (Heuristic) We say $F$ is a *fundamental solution* (or *Green's function*) for the differential operator

$$D := \frac{d^n}{dx^n} + c_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + c_0$$

if we have

$$D(F) = \delta_0.$$

Since the delta function is a bit mysterious, it can be helpful to think in terms of Fourier transforms:

**Definition.** (Heuristic) We say $F$ is a *fundamental solution* (or *Green's function*) for the differential operator $D$ above if we have

$$\widehat{D(F)}(\xi) = 1$$

for all $\xi \in \mathbb{R}$.

We can solve this equation for $\hat{F}(\xi)$ as above, obtaining

$$\hat{F}(\xi) = \frac{1}{(2\pi i\xi)^n + c_{n-1}(2\pi i\xi)^{n-1} + \cdots + c_0},$$

and reconstruct $F$ from the Fourier inversion formula (this will typically involve integrals that cannot be evaluated using elementary functions).

**Problem.** Show heuristically that if $D$ is the above differential operator, $g$ is any inhomogeneous term, and $F$ is a fundamental solution for $D$, then we have

$$D(F * g) = g,$$

i.e. the convolution $f := F * g$ solves the inhomogeneous equation $D(f) = g$.

*Hint:* Show that both sides have the same Fourier transform; start by using a previous problem to write $\widehat{D(F * g)}$ as a polynomial times $\widehat{F * g}$.

**Remark.** Fundamental solutions or Green's functions are often useful in solving boundary value problems, which we will not discuss. For instance, rather than looking for a locally-defined solution to a second-order ODE with prescribed value and first derivative at a point, one can look for a solution to the same ODE on a closed interval $[a, b]$ with prescribed values at both $a$ and $b$. Constructing a Green's function satisfying suitable properties is one way to solve such problems.