MATH 425b Homework 10

Qilin Ye

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Problem 1

Prove that all norms on a finite-dimensional vector space are equivalent.

Proof. Let V be a vector space with $\beta = \{e_1, \dots, e_n\}$ a basis for it. Define

$$\|x\|_\beta = \left(\sum_{i=1}^n |\alpha_i|^2\right)^{1/2} \quad \text{where } x = \sum_{i=1}^n \alpha_i e_i.$$

It immediately follows that $\|\cdot\|_\beta$ defines a norm on V. Furthermore, $(V, \|\cdot\|_\beta) \equiv \mathbb{R}^n$ (isometrically isomorphic). To see this, consider $T : \mathbb{R}^n \to V$ defined by

$$T(\alpha_1, \dots, \alpha_n) := \sum_{i=1}^n \alpha_i e_i.$$

Since β is a basis of V, each $x \in V$ is uniquely represented by this basis. Hence T is linear and bijective. In addition,

$$\|T(\alpha_1, \dots, \alpha_n)\|_\beta = \left(\sum_{i=1}^n |\alpha_i|^2\right)^{1/2} = |(\alpha_1, \dots, \alpha_n)|_{\text{st}}.$$

(The RHS is the standard Euclidean norm on $\mathbb{R}^n$.) Therefore we have used β to identify V with $\mathbb{R}^n$.
Now, let $\|\cdot\|$ be any other norm. By Cauchy Schwarz, we have

$$\|x\| = \left\|\sum_{i=1}^n \alpha_i e_i\right\| \leqslant \sum_{i=1}^n |\alpha_i| \|e_i\| \leqslant \underbrace{\left(\sum_{i=1}^n |\alpha_i|^2\right)^{\frac{1}{2}}}_{\|x\|_\beta} \underbrace{\left(\sum_{i=1}^n \|e_i\|^2\right)^{\frac{1}{2}}}_{:= c_2 \text{ a constant}} = c_2 \|x\|_\beta. \tag{1}$$

Now it remains to show that there exists a $c_1 > 0$ such that $c_1 \|x\|_\beta \leqslant \|x\|$. We first show that the mapping $x \mapsto \|x\|$ is continuous with respect to $\|\cdot\|_\beta$: indeed, by triangle inequality we have

$$\begin{cases} \|x\| \leqslant \|x - y\| + \|y\| \\ \|y\| \leqslant \|x\| + \|y - x\| \end{cases} \implies \begin{cases} \|x\| - \|y\| \leqslant \|x - y\| \\ \|y\| - \|x\| \leqslant \|y - x\| \end{cases} \implies |\|x\| - \|y\|| \leqslant \|x - y\| \leqslant c_2 \|x - y\|_\beta.$$

Now consider the unit sphere with respect to $\|\cdot\|_\beta$:

$$\mathcal{S} := \{v \in V : \|v\|_\beta = 1\}.$$

Certainly, by the isometry to $\mathbb{R}^n$ and Heine-Borel theorem, $\mathcal{S}$ is compact. Therefore $\|x\|$ is bounded on $\mathcal{S}$ and it also attain its bounds. In particular, the lower bound $c_1 > 0$ because it is the norm of some $x' \in \mathcal{S}$ (so $\|x'\| \neq 0$

or otherwise $x' = 0 \notin \mathcal{S}$). For any arbitrary (nonzero) $x \in V$, we have

$$c_1 \leqslant \left\| \frac{x}{\|x\|_\beta} \right\| = \frac{\|x\|}{\|x\|_\beta} \implies c_1\|x\|_\beta \leqslant \|x\|. \tag{2}$$

Combining (1) and (2) we see that any arbitrary $\|\cdot\|$ is equivalent to $\|\cdot\|_\beta$. Now, let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on V. We know both are equivalent to $\|\cdot\|_\beta$. Therefore there exist $c_1, c_2, d_1, d_2 > 0$ such that

$$\begin{cases} c_1\|x\|_\beta \leqslant \|x\|_1 \leqslant c_2\|x\|_\beta \\ d_1\|x\|_\beta \leqslant \|x\|_2 \leqslant d_2\|x\|_\beta \end{cases} \implies \frac{c_1}{d_2}\|x\|_2 \leqslant \|x\|_1 \leqslant \frac{c_2}{d_1}\|x\|_2.$$

Therefore $\|\cdot\|_1, \|\cdot\|_2$ are equivalent, and the claim follows. □

Problem 2

Show heuristically that differentiating a function f corresponds to multiplying its Fourier transform $\hat{f}(\xi)$ by the linear function $2\pi i\xi$, i.e., that

$$\widehat{f'} = 2\pi i\xi\hat{f}(\xi).$$

Proof. With the assumptions provided by the hint,

$$\begin{aligned} \widehat{f'}(\xi) &= \int_{-\infty}^{\infty} f'(\tilde{x})e^{-2\pi i\xi\tilde{x}}\,\mathrm{d}\tilde{x} \\ &= f(\tilde{x})e^{-2\pi i\xi\tilde{x}}\Big|_{\tilde{x}=-\infty}^{\infty} - (-2\pi i\xi)\int_{-\infty}^{\infty} f(\tilde{x})e^{-2\pi i\xi\tilde{x}}\,\mathrm{d}\tilde{x} \\ &= 2\pi i\xi\hat{f}(\xi). \end{aligned}$$

$$\begin{bmatrix} u = e^{-2\pi i\xi x} & \mathrm{d}u = -2\pi i\xi e^{-2\pi i\xi x}\mathrm{d}x \\ \mathrm{d}v = f'(x)\mathrm{d}x & v = f(x) \end{bmatrix}$$

□

Problem 3

Show heuristically that, for sufficiently nice $f : \mathbb{R} \to \mathbb{C}$, if

$$g := \frac{\mathrm{d}^n f}{\mathrm{d}x^n}f + c_{n-1}\frac{\mathrm{d}^{n-1}f}{\mathrm{d}x^{n-1}} + \cdots + c_0 f,$$

then

$$\hat{g}(\xi) = \left[(2\pi i\xi)^n + c_{n-1}(2\pi i\xi)^{n-1} + \cdots + c_0\right]\hat{f}(\xi).$$

Proof. By applying the result from the previous problem inductively, we have

$$\widehat{f^{(k)}}(\xi) = (2\pi i\xi)^k\hat{f}(\xi).$$

It is also clear that $\widehat{(cf)}(\xi) = c\hat{f}(\xi)$. The claim the follows. □

Problem 4

Using Fourier analysis, find all solutions $f : \mathbb{R} \to \mathbb{C}$ to the differential equation

$$f''(t) + 4f'(t) + 3f(t) = e^{4\pi it}.$$

Solution

Let $g(t) \coloneqq e^{4\pi i t}$. Then

$$\begin{aligned}\hat{g}(\xi) &= \int_{-\infty}^{\infty} g(t)e^{-2\pi i \xi t}\,\mathrm{d}t = \int_{-\infty}^{\infty} e^{4\pi i t}e^{-2\pi i \xi t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} e^{2\pi i t(2-\xi)}\,\mathrm{d}t = \delta(2-\xi).\end{aligned}$$

Notice from the previous problem that

$$\left[(2\pi i\xi)^2 + 4(2\pi i \xi) + 3\right]\hat{f}(\xi) = \hat{g}(\xi).$$

Therefore, we can rewrite $\hat{f}(\xi)$ and compute $f(x)$ from $\hat{f}(\xi)$ using the *Fourier inversion formula*:

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i \xi x}\,\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} \delta(2-\xi)\frac{e^{2\pi i \xi x}}{(2\pi i\xi)^2 + 4(2\pi i\xi) + 3}\,\mathrm{d}\xi \\ &= \left.\frac{e^{2\pi i \xi x}}{(2\pi i\xi)^2 + 4(2\pi i\xi) + 3}\right|_{\xi=2} \\ &= \frac{e^{4\pi i x}}{-16\pi^2 + 16\pi i + 3}.\end{aligned}$$

It is easy to check that this indeed solves the inhomogeneous part:

$$f''(t) + 4f'(t) + 3f(t) = \frac{1}{-16\pi^2 + 16\pi i + 3}(-16\pi^2 e^{4\pi i x} + 16\pi i e^{4\pi i x} + 3e^{4\pi i x}) = e^{4\pi i x}.$$

Claim: the general solution is of form

$$f(t) = c_1 e^{-t} + c_2 e^{-3t} + \frac{e^{4\pi i x}}{-16\pi^2 + 16\pi i + 3}.$$

In other words, any solution to the homogeneous DE $f''9t) + 4f'(t) + 3f(t) = 0$ is a linear combination of e^{-t} and e^{-3t}. To see this, we first apply reduction of order and rewrite the DE as

$$\begin{bmatrix} f'(t) \\ f''(t)\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4\end{bmatrix}\begin{bmatrix} f(t) \\ f'(t)\end{bmatrix}. \tag{1}$$

It follows that f solves the DE if and only if the above vector equation holds. By inspection, we can immediately tell that e^{-t} and e^{-3t} are two solutions to the DE. Notice that

$$\begin{bmatrix} e^{-0} \\ -e^{-0}\end{bmatrix} = \begin{bmatrix} 1 \\ -1\end{bmatrix} \text{ and } \begin{bmatrix} e^{-3\cdot 0} \\ -3e^{-3\cdot 0}\end{bmatrix} = \begin{bmatrix} 1 \\ -3\end{bmatrix}$$

are linearly independent, so they span $\mathbb{R}^2$. Now we re-write (1) as a vector function $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F[(u(t), v(t)]^T = \begin{bmatrix} 0 & 1 \\ -3 & -4\end{bmatrix}\begin{bmatrix} u(t) \\ v(t)\end{bmatrix}.$$

Clearly F is locally Lipschitz as all entries in the matrix are finite. Now we invoke the Picard-Lindelöf theorem: given any initial condition $u(0) = u_0, v(0) = v_0$, there exists (a unique) $\gamma(t) = [\gamma_u(t), \gamma_v(t)]^T$ satisfying (1). On the other hand, $\gamma(t) \in \mathbb{R}^2 = \text{span}\{[1,-1]^T, [1,-3]^T\}$. Therefore $\gamma_u(t)$ is a linear combination of e^{-t} and e^{-3t}[!] This means precisely that any solution to the homogeneous system is of form $c_1 e^{-t} + c_2 e^{-3t}$. $\square$

Problem 5

Show heuristically that if D is the above differential operator, g any inhomogeneous term, and F a fundamental solution for D, then

$$D(F * g) = g,$$

i.e., the convolution $f := F * g$ solves the inhomogeneous equation $D(f) = g$.

Proof. Let $f := F * g$. Notice that the Fourier coefficient $\widehat{D(f)}(\xi)$ (by problem 3) is

$$\left[(2\pi i\xi)^n + c_{n-1}(2\pi i\xi)^{n-1} + \cdots + c_0\right]\hat{f}(\xi) = [\ldots]\hat{F}\hat{g} = [\ldots] \cdot \frac{\hat{g}}{[\ldots]} = \hat{g}.$$

(The first equation directly follows from the fact that $\widehat{F * g} = \hat{F}\hat{g}$.) □