MATH 425b Homework 11

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April 9, 2021

Problem 1

Let $\mathcal{F}(X)$ denote the vector space of *suitably nice* $\mathbb{C}$-valued functions on X. Show heuristically that the following commutes when the "F.T." arrows point to the right:

$$\begin{array}{ccc} \mathcal{F}(\mathbb{R}) & \xleftarrow{\quad} \text{F.T.} \xrightarrow{\quad} & \mathcal{F}(\mathbb{R}) \\ \Big\downarrow \text{resum} & & \Big\downarrow \text{sample} \\ \mathcal{F}(\mathbb{R}/\mathbb{Z}) & \xleftarrow{\quad} \text{F.T.} \xrightarrow{\quad} & \mathcal{F}(\mathbb{Z}) \end{array}$$

i.e., for $f \in \mathcal{F}(\mathbb{R})$ we have

$$\text{sample}(\text{F.T.}(f)) = \text{F.T.}(\text{resum}(f)). \tag{1}$$

Also show heuristically that the diagram commutes when the "F.T." arrows point to the left, i.e., for $f \in \mathcal{F}(\mathbb{R})$,

$$\text{resum}(\text{F.T.}(f)) = \text{F.T.}(\text{sample}(f)). \tag{2}$$

Proof. We first show (1), i.e., commutativity when the F.T. arrows point to the right. For $n \in \mathbb{Z}$, the n^{th} Fourier coefficient of $\text{resum}(f)$ is

$$\begin{aligned} \widehat{\text{resum}(f)}(n) &= \int_0^1 e^{-2\pi i n\theta}(\text{resum}(f))(\theta)\,\mathrm{d}\theta \\ &= \int_0^1 e^{-2\pi i n\theta}\sum_{k=-\infty}^{\infty} f(\theta+k)\,\mathrm{d}\theta \\ &= \sum_{k=-\infty}^{\infty}\int_0^1 e^{-2\pi i n\theta} f(\theta+k)\,\mathrm{d}\theta \\ &= \sum_{k=-\infty}^{\infty}\int_k^{k+1} e^{-2\pi i n u} f(u)\,\mathrm{d}u \\ &= \int_{-\infty}^{\infty} e^{-2\pi i n u} f(u)\,\mathrm{d}u = \hat{f}(n). \end{aligned}$$

The LHS gives exactly the same thing when evaluated at an integer n. Therefore (1) holds.
Now we show (2), i.e., commutativity when the F.T. arrows point to the left. Define a reflection function

$$f \mapsto \text{refl}(f) \text{ by } (\text{refl}(f)) = (x) - f(-x).$$

Notice that

$$(\text{F.T.}(f))(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i \xi x}\,\mathrm{d}x \qquad \text{and} \qquad f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i \xi x}\,\mathrm{d}\xi$$

which give

$$(\text{F.T.}^2(f))(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-2\pi i \xi x}\, d\xi = f(-x).$$

Therefore $(\text{F.T.})^2 = \text{refl}$. Thus, if we apply F.T. one more time to both sides of (2), the RHS simply becomes $\text{refl}(\text{sample}(f)) : n \mapsto f(-n)$. The LHS, on the other hand, becomes

$$\begin{aligned} \text{F.T.} \circ \text{resum} \circ \text{F.T.} &= [\text{F.T.} \circ \text{resum}] \circ \text{F.T.} \\ &= \text{sample} \circ \text{F.T.} \circ \text{F.T.} && \text{(by the previous part)} \\ &= \text{sample} \circ \text{refl}. \end{aligned}$$

It is clear that the $\text{sample} \circ \text{refl} = \text{refl} \circ \text{sample}$: for $n \in \mathbb{Z}$,

$$[\text{sample} \circ \text{refl}(f)](n) = [\text{sample}(f)](-n) = f(-n) = [\text{refl} \circ \text{sample}(f)](n).$$

Thus (2) holds and we are done. □

Problem 2

Heuristically deduce the *Poisson summation formula*: for suitably nice $f \in \mathcal{F}(\mathbb{R})$, we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

Proof. Recall from the first problem that

$$\text{resum}(\text{F.T.}(f)) = \text{F.T.}(\text{sample}(f)).$$

Let $\xi = 0$. The RHS then becomes

$$\sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi} = \sum_{n=-\infty}^{\infty} f(n).$$

On the other hand, the LHS gives

$$\sum_{n=-\infty}^{\infty} (\text{F.T.}(f))(\xi + n) = \sum_{n=-\infty}^{\infty} (\text{F.T.}(f))(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$ □

Problem 3

Let $f, g \in \mathcal{F}(\mathbb{R})$ and suppose that $\hat{f}(\xi) = \hat{g}(\xi) = 0$ unless $\xi \in (0,1)$, i.e., $\hat{f}$ and $\hat{g}$ are supported on the interior of the compact interval $[0,1]$. Assuming that $f(n) = g(n)$ for all $n \in \mathbb{Z}$, show heuristically that $f = g$.

Proof. To show $f = g$, it suffices to show $\hat{f} = \hat{g}$. Since $\hat{f} \equiv \hat{g} \equiv 0$ outside $(0,1)$ already, $\hat{f} = \hat{g}$ if and only if $\hat{f}(\xi) = \hat{g}(\xi)$ for all $\xi \in (0,1)$. Notice that, for $\xi \in (0,1)$,

$$\hat{f}(\xi) = \hat{g}(\xi) \iff \sum_{n=-\infty}^{\infty} \hat{f}(\xi + n) = \hat{f}(\xi) = \hat{g}(\xi) = \sum_{n=-\infty}^{\infty} \hat{g}(\xi + n).$$

Thus, $f = g$ if and only if $\hat{f} = \hat{g}$ if and only if $\text{resum}(\hat{f}) = \text{resum}(\hat{g})$. Now we invoke the result shown in the first problem again. Since $\text{resum}(\hat{f}) = \text{F.T.}(\text{sample}(f))$ and likewise for g,

$$\text{resum}(\hat{f}) = \text{resum}(\hat{g}) \iff \text{F.T.}(\text{sample}(f)) = \text{F.T.}(\text{sample}(g)).$$

But we know this is true, as f and g are assumed to agree on all integer values! The claim therefore follows. $\square$

Definition

Let S be a set. Assume that for every $s \in S$ we have a vector space V_s over $\mathbb{R}$ (or $\mathbb{C}$). Define $\bigoplus_{s\in S} V_s$, the **direct sum** of the vector spaces V_s, to be the set of functions $f : S \to \bigcup_{s\in S} V_s$ such that $f(s) \in V_s$ for all $s \in S$ and $f(s) = 0$ for all but finitely many $s \in S$. For $f, g \in \bigoplus_{s\in S} V_s$, define $f + g$ by $(f + g)(s) = f(s) + g(s)$. For $c \in \mathbb{R}$, define cf by $(cf)(s) = c(f(s))$.

Problem 4

(a) Prove that $\bigoplus_{s\in s} V_s$ is closed under addition and scalar multiplication as defined above.

(b) When $S = \{s, t\}$ only has two elements, we get a vector space $V_s \oplus V_t$ from the two vector spaces V_s and V_t. Prove that $V_s \oplus V_t$ is isomorphic to the vector space of ordered pairs (v, w) where $v \in V_s, w \in V_t$, and the addition and scalar multiplication are given by $(v, w) + (v', w') = (v + v', w + w')$ and $c(v, w) = (cv, cw)$.

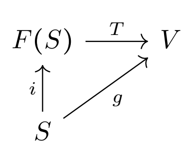
(c) If S is any set, define the **free vector space** on S to be

$$F(S) := \bigoplus_{s\in S} \mathbb{R},$$

i.e., we take each vector space V_s to be $\mathbb{R}$. We have a function $i : S \to F(S)$ sending $s \in S$ to the function $i(s) : S \to V$ by

$$i(s)(s') = \begin{cases} 1 & \text{if } s' = s \\ 0 & \text{otherwise.} \end{cases}$$

Let V be any vector space and let $g : S \to V$ be any function. Prove that there exists a unique linear transformation $T : F(S) \to V$ such that $g = T \circ i$, i.e., such that the following diagram commutes:



Proof of 4(a). Suppose f, g are in the direct sum. Then $f(s) \neq 0$ for at most finitely many $s \in S$ and the same holds for g. If $(f + g)(s) = f(s) + g(s) \neq 0$ then either $f(s) \neq 0$ or $g(s) \neq 0$ (or both), but this can only happen for finitely many $s \in S$. Therefore $f + g$ is also in the direct sum. That $cf \in \bigoplus_{s\in S} V_s$ is immediate given the linearity of f. $\square$

Proof of 4(b). Consider the map $T : V_s \oplus V_t \to V_s \times V_t$ by $T(f) = (f(s), f(t))$. Since f is well-defined, so is T. For injectivity, if $f \neq g$ then $f(x) \neq g(x)$ for <u>some</u> $x \in s \in S$. Thus either $f(s) \neq g(s)$ or $f(t) \neq g(t)$. In either case $T(f) \neq T(g)$. Therefore T is injective.

Surjectivity is trivial, and linearity follows directly from that of f. $\square$

Proof of 4(c). We first show existence. For $f \in F(S)$, define $T(f) := \sum_{s \in S} f(s)g(s)$. T is linear because

$$\begin{aligned} T(f_1 + f_2) &= \sum_{s \in S} (f_1 + f_2)(s)g(s) \\ &= \sum_{s \in S} [f_1(s) + f_2(s)]\, g(s) \\ &= \sum_{s \in S} f_1(s)g(s) + \sum_{s \in S} f_2(s) = T(f_1) + T(f_2), \end{aligned}$$

and

$$T(cf) = \sum_{s \in S} (cf)(s)g(s) = \sum_{s \in S} cf(s)g(s) = c\sum_{s \in S} f(s)g(s) = cT(f).$$

The diagram commutes because, while g sends $s \in S$ to $g(s) \in V$,

$$(T \circ i)(s) = \sum_{s' \in S} [i(s)](s')g(s') = [i(s)](s)g(s) + \sum_{s' \neq s} 0 \cdot g(s') = g(s).$$

Now we show the uniqueness of T. Suppose for some other T' we have $T \circ i = g = T' \circ i$, that is,

$$T(i(s)) = g(s) = T'(i(s)) \text{ for all } s \in S.$$

Let $f \in F(S)$ be arbitrarily chosen. Recall that $f(s) \neq 0$ for at most finitely many $s \in S$. Let $\{s_i\}_{i=1}^n$ be the set of all elements of S on which f does not vanish. It follows that

$$f = \sum_{i=1}^n f(s_i)i(s_i).$$

Indeed, if f is defined as such, we get

$$f(s) = \sum_{i=1}^n f(s_i)[i(s_i)](s) = \begin{cases} f(s_k)[i(s_k)](s_k) = f(s_k) & \text{if } s = s_k \in \{s_i\}_{i=1}^n \\ \sum_{i=1}^n f(s_i)[i(s_i)](s) = \sum_{i=1}^n f(s_i) \cdot 0 = 0 & \text{otherwise.} \end{cases}$$

This summation recovers f precisely! Then, by linearity,

$$\begin{aligned} T(f) &= T(\sum_{i=1}^n f(s_i)i(s_i)) = \sum_{i=1}^n f(s_i)T(i(s_i)) \\ &= \sum_{i=1}^n f(s_i)g(s_i) = \sum_{i=1}^n f(s_i)T'(i(s_i)) \\ &= T'(\sum_{i=1}^n f(s_i)i(s_I)) = T'(f), \end{aligned}$$

and we are done. □