HOMEWORK, WEEK 11

This assignment is due Monday, April 5. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

(1) We've looked at Fourier series for periodic functions and Fourier transforms for non-periodic functions; now we'll look at how they're related. As usual for our treatment of Fourier transforms, we'll work at a heuristic level.

First we consider two important operations one can perform on suitably nice functions $f : \mathbb{R} \to \mathbb{C}$:

Definition. If $f : \mathbb{R} \to \mathbb{C}$ is a suitably nice function, define the *periodic resummation* $\mathrm{resum}(f)$ of f by

$$\mathrm{resum}(f)(x) := \sum_{n=-\infty}^{\infty} f(x+n),$$

a periodic function from $\mathbb{R}$ to $\mathbb{C}$ with period 1.

As on a previous homework set, we can view $\mathrm{resum}(f)$ as a function on the unit circle S^1; equivalently, we can view $\mathrm{resum}(f)$ as a function on $\mathbb{R}/\mathbb{Z}$, the set of equivalence classes of real numbers under the equivalence relation $r_1 \sim r_2$ if $r_1 - r_2 \in \mathbb{Z}$.

Definition. If $f : \mathbb{R} \to \mathbb{C}$ is a suitably nice function, define the *sample sequence* $\mathrm{sample}(f)$ by

$$\mathrm{sample}(f)(n) := f(n),$$

a function from $\mathbb{Z}$ to $\mathbb{C}$ (this is just the restriction of f to $\mathbb{Z} \subset \mathbb{R}$).

For $X \in \{\mathbb{R}, \mathbb{Z}, \mathbb{R}/\mathbb{Z}\}$, write $\mathrm{Fun}(X)$ for the vector space of suitably nice $\mathbb{C}$-valued functions on X. We have a diagram of linear transformations

$$\begin{array}{ccc} \mathrm{Fun}(\mathbb{R}) & \xleftrightarrow{\text{F.T.}} & \mathrm{Fun}(\mathbb{R}) \\ {\scriptstyle\text{resum}}\downarrow & & \downarrow{\scriptstyle\text{sample}} \\ \mathrm{Fun}(\mathbb{R}/\mathbb{Z}) & \xleftrightarrow[\text{F.T.}]{} & \mathrm{Fun}(\mathbb{Z}) \end{array} \quad ;$$

where "F.T." means Fourier transform. The arrow pointing to the right on the bottom edge sends $f \in \mathrm{Fun}(\mathbb{R}/\mathbb{Z})$ to its sequence of Fourier coefficients. The arrow pointing to the left on the bottom edge sends $(a_n)_{n=-\infty}^{\infty} \in \mathrm{Fun}(\mathbb{Z})$ to $(\mathrm{F.T.}((a_n)_{n=-\infty}^{\infty}) \in \mathrm{Fun}(\mathbb{R}/\mathbb{Z})$ where

$$(\mathrm{F.T.}((a_n)_{n=-\infty}^{\infty}))(x) := \sum_{n=-\infty}^{\infty} a_n e^{-2\pi i n x},$$

which is the Fourier series built from the coefficients $(a_n)_{n=-\infty}^{\infty}$, but evaluated at $-x$ rather than at x.

Remark. This last instance of the "Fourier transform" is typically called the *discrete-time Fourier transform* or DTFT. With this language, $(a_n)_{n=-\infty}^{\infty}$ is typically thought of as being sample(f) where f is some continuous-time signal (e.g. air pressure in a given direction at a given point, when doing acoustics). If you wanted to take the Fourier transform $\mathrm{F.T.}(f) = \hat{f}$ of f but only had access to the sample values sample(f), you could take the DTFT of sample(f) and hope you can recover the information want about $\hat{f}$.

Problem. Show heuristically that the above diagram commutes when the "F.T." arrows point to the right, i.e. for $f \in \mathrm{Fun}(\mathbb{R})$, we have

$$\mathrm{sample}(\mathrm{F.T.}(f)) = \mathrm{F.T.}(\mathrm{resum}(f)).$$

Similarly, show heuristically that the diagram commutes when the "F.T." arrows point to the left, i.e. for $f \in \mathrm{Fun}(\mathbb{R})$, we have

$$\mathrm{resum}(\mathrm{F.T.}(f)) = \mathrm{F.T.}(\mathrm{sample}(f)).$$

Hint: For the first statement, write out the definition of F.T.(resum(f)); note that this instance of F.T. is extracting Fourier coefficients of a periodic function, so it should involve an integral $\int_0^1$ rather than $\int_{-\infty}^{\infty}$. Integrating by substitution, try to write the result as a sum of integrals $\int_n^{n+1}$ that you can assemble into an integral $\int_{-\infty}^{\infty}$.

For the second statement, it's easiest to proceed formally: show that when applying F.T. to both sides, you get the same result using the first statement. It's useful that doing F.T. twice can be written as $f \mapsto \mathrm{refl}(f)$, where $(\mathrm{refl}(f))(x) = f(-x)$ in all cases.

Remark. It follows that if we take the DTFT of sample(f) for some $f \in \mathrm{Fun}(\mathbb{R})$, the result is the same as applying periodic resummation to the Fourier transform $\mathrm{F.T.}(f) = \hat{f}$. If f represents some continuous-time signal and we would like to recover $\hat{f}$ from the sample sequence sample(f), we might hope that we are in a case where the periodic resummation operation applied to $\hat{f}$ "loses no data." Then we should be able to use resum$(\hat{f})$ to deduce $\hat{f}$ itself.

Remark. Recall that the Fourier transform comes from the Pontryagin self-duality of $\mathbb{R}$: we have $\hat{\mathbb{R}} \cong \mathbb{R}$. Similarly, Fourier series come from the Pontryagin duality $\hat{\mathbb{Z}} = \mathbb{R}/\mathbb{Z}$. These instances of duality are related: $\mathbb{Z}$ is a subgroup of $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ is a quotient of $\mathbb{R}$. In general, let G be a locally compact topological abelian group and let H be a closed subgroup of G. We have isomorphisms $\hat{H} \cong \hat{G}/\operatorname{ann}(H)$ and $\widehat{G/H} \cong \operatorname{ann}(H)$, where $\operatorname{ann}(H)$ is the "annihilator of H" (this makes sense if $\hat{G}$ is defined to be the group of characters, i.e. homomorphisms from G into S^1). For $G = \mathbb{R}$ and $H = \mathbb{Z} \subset \mathbb{R}$, an element $(x \mapsto e^{2\pi i \xi x})$ of $\hat{\mathbb{R}} \cong \mathbb{R}$ annihilates $H = \mathbb{Z}$ if and only if ξ is an integer.

There are maps sample : $\mathrm{Fun}(G) \to \mathrm{Fun}(H)$ (just restriction of functions to H) and resum : $\mathrm{Fun}(G) \to \mathrm{Fun}(G/H)$. With appropriate choices of measure on all the groups in question, the squares

$$\begin{array}{ccc} \mathrm{Fun}(G) & \xleftrightarrow{\text{F.T.}} & \mathrm{Fun}(\hat{G}) \\ \text{resum}\downarrow & & \downarrow\text{sample} \\ \mathrm{Fun}(G/H) & \xleftrightarrow[\text{F.T}]{} & \mathrm{Fun}(\mathrm{ann}(H)) \end{array}$$

commute, where the F.T. arrows point either to the right or to the left.

Remark. Fourier transforms require integrals (typically Lebesgue), which require measures. The key result is that for locally compact Hausdorff topological groups, there exists a unique translation-invariant measure up to rescaling, called *Haar measure*. We are always using Haar measure on all groups in sight; the only issue is fixing the scaling factors. It's not hard to see that scaling the Haar measure on $\hat{G}$ relative to that on G will introduce a scaling factor in the Fourier inversion formula; given a choice of Haar measure on G, there exists a unique choice of Haar measure on $\hat{G}$ such that the Fourier inversion formula has no scaling factor.

In our case, Haar measure on $\mathbb{R}$ is Lebesgue measure up to scaling; we take the standard Lebesgue measure. However, we need to be careful when identifying $\hat{\mathbb{R}}$ (with its Haar measure defined so that the Fourier transform has no scaling factor) with ($\mathbb{R}$, usual Lebesgue measure). The measure-preserving identification here sends $\xi \in \mathbb{R}$ to $(x \mapsto e^{2\pi i \xi x})$ in $\hat{\mathbb{R}}$. This is the reason I use $e^{-2\pi i \xi x}$ rather than $e^{i\xi x}$ in the Fourier transform, and why I don't have a factor of $\frac{1}{2\pi}$ in the Fourier inversion formula.

(2) Using the result of the previous problem, heuristically deduce the *Poisson summation formula*: for suitably nice $f \in \mathrm{Fun}(\mathbb{R})$, we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

Hint: This is a bit of magic. The DTFT gives you sums from $-\infty$ to ∞, so when you see a sum from $-\infty$ to ∞, you can try thinking of it as coming from a DTFT. On the left side of the equation, we have a sum from $-\infty$ to ∞ involving f sampled at integer points. If you applied the DTFT to the sequence $(f(n))_{n=-\infty}^{\infty}$, the definition would give you the function

$$\xi \mapsto \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi},$$

and now you can notice that the left side of the Poisson summation formula is the evaluation of this function at $\xi = 0$. From here, the commutative squares should give you what you need.

(3) Another consequence of the above commutative squares is the Shannon–Nyquist sampling theorem, which is a fundamental result in signal processing whose applications you're already very familiar with if you listen to music electronically. Suppose $f \in \mathrm{Fun}(\mathbb{R})$ represents air pressure over time; you can only sample $f(t)$ at discrete sample points t, but you'd like enough data to recover f uniquely (so you can play it back accurately). For an arbitrary function f, this is (of course) impossible, but it turns out to be possible given a suitable assumption on the Fourier transform $\hat{f}$ of f. Specifically, you want $\hat{f}$ to be compactly supported (i.e. zero outside a compact interval of $\mathbb{R}$); if this holds, the original function f is called "band-limited."

Problem. Let $f, g \in \mathrm{Fun}(\mathbb{R})$ and suppose that $\hat{f}(\xi) = \hat{g}(\xi) = 0$ unless $\xi \in (0,1)$ (i.e. $\hat{f}$ and $\hat{g}$ are supported on the interior of the compact interval $[0,1]$). Assuming that $f(n) = g(n)$ for all integers n, show heuristically that $f = g$.

Hint: It suffices to show that $\hat{f} = \hat{g}$; show it suffices to show that $\mathrm{resum}(\hat{f}) = \mathrm{resum}(\hat{g})$ given the assumptions, and use a commutative square from above.

Remark. The above problem was set up to avoid extra constants that complicate the computations, but realistically one samples $f \in \mathrm{Fun}(\mathbb{R})$ on some subgroup $T\mathbb{Z} \subset \mathbb{R}$ (for some $T > 0$ that might not be $T = 1$). A character $\xi \mapsto e^{2\pi i \xi x}$ on $\mathbb{R}$ annihilates $T\mathbb{Z}$ if and only if $\xi \in (1/T)\mathbb{Z}$, so to make the above squares work, the bottom-left corner should be $\frac{\mathbb{R}}{(1/T)\mathbb{Z}}$. Thus, if $\hat{f}$ and $\hat{g}$ are supported on the interior of $[0, 1/T]$, and we have $f(tn) = g(tn)$ for all $n \in \mathbb{Z}$, then $f = g$ everywhere.

Furthermore, it is not so realistic to ask that $\hat{f}$ and $\hat{g}$ have no negative Fourier coefficients, although it is often realistic to ask that the Fourier coefficients are bounded above in absolute value (values of $\hat{f}(\xi)$ at large $|\xi|$ correspond acoustically to very high-pitch components of the sound, which can be thrown away if they are above the range of human hearing). If $\hat{f}$ and $\hat{g}$ are supported on the interior of $[-1/(2T), 1/(2T)]$, it still follows that $\mathrm{resum}(\hat{f})$ and $\mathrm{resum}(\hat{g})$ determine $\hat{f}$ and $\hat{g}$, so the argument goes through. Thus, the usual statement of the theorem is that a sample interval of $1/(2B)$ (seconds in between samples) is sufficient assuming f is band-limited to $[-B, B]$ (B measured in hertz, i.e. cycles per second; we are setting $B = 1/(2T)$ so $T = 1/(2B)$).

One often works in terms of the sample rate $f_s = 2B$, the number of samples per second, instead of the sample interval $1/f_s = 1/(2B)$. For a sample rate of f_s to be sufficient, $\hat{f}$ should be supported on $[-f_s/2, f_s/2]$. Audio signals are often sampled at $f_s = 44100$ hertz; an assumption that $\hat{f}$ is band-limited to $[-22050, 22050]$ is workable because the range of human hearing is about 20 hertz to 20000 hertz.

(4) Now we'll begin studying tensors. Besides being crucial for differential forms, tensors are important in many areas of math, science, and even machine learning. The perspective I want to take is that of "operations on vector spaces," especially the tensor product of vector spaces. We know about operations *in* vector spaces; we can add two vectors and get another vector. It turns out that we can also add two vector spaces and get another vector space (the direct sum), such that the dimensions add.

The tensor product will be a way of multiplying two vector spaces to get another vector space, such that the dimensions multiply.

Definition. Let S be any set. Assume that for every $s \in S$, we have a vector space V_s over $\mathbb{R}$ (it would be just as easy to work over $\mathbb{C}$). Define

$$\bigoplus_{s \in S} V_s,$$

the *direct sum* of the vector spaces V_s, to be the set of functions $f : S \to \cup_{s' \in S} V_{s'}$ such that $f(s) \in V_s$ for all $s \in S$, and such that $f(s) = 0$ for all but finitely many $s \in S$. (Note that the union is a "disjoint union," i.e. the vector spaces $V_{s'}$ are not given as subsets of a larger set *a priori*.)

For $f, g \in \oplus_{s \in S} V_s$, define $f + g$ by $(f + g)(s) = f(s) + g(s)$. For $c \in \mathbb{R}$, define cf by $(cf)(s) = c(f(s))$.

Remark. You can think of $\oplus_{s \in S} V_s$ as the vector space of all formal linear combinations $c_1 v_1 + \cdots + c_n v_n$ where for $1 \leq i \leq n$, we have $v_i \in V_{s_i}$ for some $s_1 \neq \cdots \neq s_n$. The linear combinations are finite sums because we require $f(s) = 0$ for all but finitely many $s \in S$. If we did not impose this condition, we would have the direct product $\Pi_{s \in S} V_s$, which cannot always be viewed in terms of finite linear combinations.

(a) Prove that $\oplus_{s \in S} V_s$ is closed under the above addition and scalar multiplication operations.

Hint: This is very short- you just need to check the "vanishes at all but finitely many s" property for sums and scalar multiples.

One could also check the vector space axioms for $\oplus_{s \in S} V_s$, and I recommend doing this in your head, but it would be tedious to write out and it's not required.

(b) In particular, when S has two elements $S = \{s, t\}$, we get a vector space $V_s \oplus V_t$ from two given vector spaces V_s and V_t. Prove that $V_s \oplus V_t$ is isomorphic to the vector space of ordered pairs (v, w) where $v \in V_s$, $w \in V_t$ (i.e. the Cartesian product $V_s \times V_t$), and the addition and scalar multiplication are given by $(v, w) + (v', w') = (v + v', w + w')$ and $c(v, w) = (cv, cw)$.

Hint: You can define a linear map sending a function $f \in V_s \oplus V_t$ as defined above to the ordered pair $(f(s), f(t))$. Check that this map is linear and bijective. Again, this proof should be short.

Remark. From a basis $\{e_\alpha\}$ for V_s and a basis f_β for V_t, one gets elements $(e_\alpha, 0)$ and $(0, f_\beta)$ of $V_s \oplus V_t$. One can check that these elements form a basis for $V_s \oplus V_t$. In particular, if V and W are vector spaces of dimension n and m respectively, then $V \oplus W$ has dimension $n + m$. Soon we will define a vector space $V \otimes W$, the tensor product of V and W, whose dimension is nm.

(c) If S is any set, define the *free vector space on S* to be

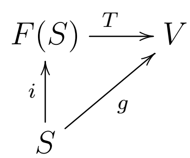
$$F(S) = \bigoplus_{s \in S} \mathbb{R},$$

i.e. we take each vector space V_s to be $\mathbb{R}$. We have a function $i : S \to F(S)$ sending $s \in S$ to the function $i(s) : S \to V$ defined by

$$i(s)(s') = \begin{cases} 1 & \text{if } s' = s \\ 0 & \text{otherwise.} \end{cases}$$

Remark. You can think of $F(S)$ as the vector space of formal linear combinations of elements of S; the elements of S form a basis for $F(S)$ by definition. Equivalently, $F(S)$ is a way to construct a vector space having a given set S as a basis.

Problem. Let V be any vector space and let $g : S \to V$ be any function. Prove that there exists a unique linear transformation $T : F(S) \to V$ such that $g = T \circ i$, i.e. such that the following diagram commutes:



Hint: For $f \in F(S)$, define $T(f)$ to be the element of V given by $\sum_{s \in S} f(s)g(s)$ (note that $f(s)$ is a real number and $g(s)$ is a vector in V). The sum is finite since $f(s) = 0$ for all but finitely many s. Check (briefly) that T is linear and that the diagram commutes.

For uniqueness, if T and T' are two linear maps that the diagram commutes, you want to show that for each $f \in F(S)$ we have $T(f) = T'(f)$. To do this, write f as a linear combination of elements $i(s) \in F(S)$ for finitely many $s \in S$ (this is possible since $f(s) = 0$ for all but finitely many s). Deduce uniqueness from the (assumed) fact that $T(i(s)) = g(s) = T'(i(s))$ for all s.

Remark. As mentioned above, $F(S)$ can be thought of as the vector space of formal linear combinations of elements of S. This problem shows that any function g from a set S to a vector space V can be "extended linearly," in a unique way, to a *linear* transformation defined on the vector space of formal linear combinations of elements of S.