MATH 425b Homework 12

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April 12, 2021

Problem 1

Let V be a vector space and let W be a subspace of V. For two elements $\alpha, \beta \in v/w$, define $\alpha + \beta$ to be $[v_1 + v_2]$ where $\alpha = [v_1]$ and $\beta = [v_2]$. Also, for $c \in \mathbb{R}$ define $c\alpha$ to be $[cv]$ where $\alpha = [v]$. Prove that these notions are well-defined.

Proof. For addition, we need to show that if $a_1 \sim a_2$ and $b_1 \sim b_2$ then $a_1 + b_1 \sim a_2 + b_2$. Indeed, if our assumption holds then $1 - a_2 \in W$ and $b_1 - b_2 \in W$. Since W is a subspace, $(a_1 - a_2) + (b_1 - b_2) = (a_1 + b_1) - (a_2 + b_2) \in W$, so $a_1 + b_1 \sim a_2 + b_2$, as desired. Likewise, if $a_1 - a_2 \in W$ then so is $c(a_1 - a_2) = ca_1 - ca_2$ and thus the notion of multiplication is also well-defined. □

Problem 2

Given the setup of the previous problem, let Z be another vector space and let $f : V \to Z$ be a linear map such that $f(w) = 0$ for all $w \in W$. Let $p : V \to V/W$ denote the linear map sending v to $[v]$. Prove that there exists a unique linear map $g : V/W \to Z$ such that $f = g \circ p$.

Proof. Consider $g : [v] \mapsto f(v)$. This is well-defined because if $[v_1] = [v_2]$ then $g([v_1]) = f(v_1) = f(v_2) = g([v_2])$ as the linearity of f implies $f(v_1) - f(v_2) = f(v_1 - v_2) = 0$ (since $v_1 - v_2 \in W$). For linearity, since $[v_1] + [v_2] = [v_1 + v_2]$ by definition and f is linear, we have $g([v_1] + [v_2]) = g([v_1 + v_2]) = f(v_1 + v_2) = f(v_1) + f(v_2) = g([v_1]) + g([v_2])$. Therefore g is well-defined and linear.

For uniqueness, by definition we know any $[v] \in V/W$ equals to $p(v)$. Therefore, if $g \circ p = \tilde{g} \circ p$, i.e., $g(p(x)) = f(x) = \tilde{g}(p(x))$ for all $x \in V$, it must in particular hold for $p(v)$. Thus $g([v]) = \tilde{g}([v])$. Since $[v] \in V/W$ is chosen arbitrarily we conclude that $g = \tilde{g}$, i.e., the existence of g is unique. □

Problem 3

Show that the map $\pi : V \times W \to V \oplus W$ sending (v, w) to $v \oplus w$ is bilinear.

Proof. We want to show that $\pi(u + cv, w)$ and $\pi(u, w) + c\pi(v, w)$ are in the same equivalence class in $V \oplus W$ for all $u, v \in V, w \in W$, and $c \in \mathbb{R}$. Indeed,

$$\begin{aligned}(u + cv, w) - (u, w) - c(vw) &= (u + cv, w) - (u, w) - (cv, w) + (u, w) + (cv, w) - (u, w) - c(v, w) \\ &= [(u + cv, w) - (u, w) - (cv, w)] + [(cv, w) - c(v, w)] \in \operatorname{span}(S).\end{aligned}$$

Therefore π is linear with respect to the first argument, and one can show analogously that π is also linear with respect to the second argument. Hence π is bilinear. □

Problem 4

Let V, W, and Z be vector spaces over $\mathbb{R}$ and let $f : V \times W \to Z$ be a bilinear map. Show that there exists a unique linear map $g : V \oplus W \to Z$ such that $f = g \circ \pi$. This is called the *universal property* of tensor products.

Proof. From HW11 we know that there exists a unique linear transformation $T : F(V \times W) \to Z$ such that $f = T \circ i$, where $i : V \times W \to F(V \times W)$ sends (v, w) to $i(v, w)$ the indicator function. Suppose we have some element $s \in S$ — it may be of form $i(v, w) + i(v', w) - i(v + v', w)$ for some $v, v' \in V$ and $w \in W$. (This is the "first form" given in HW12; there are three more.) It follows from linearity of T, f and $T \circ i = f$ that

$$\begin{aligned} T(s) &= T(i(v, w) + i(v', w) - i(v + v', w)) \\ &= T(i(v, w)) + T(i(v', w)) - T(i(v + v', w)) \\ &= f(v, w) + f(v', w) - f(v + v', w) = f(0) = 0. \end{aligned}$$

Notice that, although for elements in $F(V \times W), i(v, w) + i(v', w) \neq i(v + v', w)$, thanks to $T \circ i = f$ we are able to convert them into elements of $V \times W$ where the bilinearity of f guarantees that this expression evaluates to 0. One can show analogously that T vanishes on elements of S of the other three forms as well. By problem 2 there exists $g : F(V \times W)/S = V \oplus W \to Z$ such that $g \circ \pi = f$.

For uniqueness, suppose g and $\tilde{g}$ both satisfy the conditions, i.e., $g \circ \pi = f = \tilde{g} \circ \pi$. Then for $v \oplus w = [(v, w)] \in V \oplus W$ we have

$$g(v \oplus w) = g(\pi(v, w)) = f(v, w) = \tilde{g}(\pi(v, w)) = \tilde{g}(v \oplus w),$$

i.e., g and $\tilde{g}$ agree on all pure tensors. Then, since both are linear and any tensor must be a finite linear combination of pure tensors, we conclude that $g(x) = \tilde{g}(x)$ for all $x \in V \oplus W$, i.e., $g = \tilde{g}$. □