HOMEWORK, WEEK 13

This assignment is due Monday, April 19. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

(1) Along with tensor products, another essential operation on vector spaces is the notion of the *dual space*.

Definition. If V is a vector space over $\mathbb{R}$, its *dual space* V^* is defined to be the vector space of linear transformations from V to $\mathbb{R}$ (linear transformations into $\mathbb{R}$ are often called *linear functionals*).

If $\{e_i\}$ is a basis for V, define $e_i^* \in V^*$ by $e_i^*(e_j) = \delta_{i,j}$ (following the up/down index conventions mentioned in the previous homework, one often writes e^i instead of e_i^*).

Problem. Show that if V is finite-dimensional with basis $\{e_i\}$, then the linear functionals $\{e_i^*\}$ form a basis for V^*; in particular, V^* has the same dimension as V. Also show (using ideas from last week's homework) that if V and W are finite-dimensional with bases $\{e_i\}$ and $\{f_j\}$, then the elements $e_i \otimes f_j$ form a basis for $V \otimes W$; in particular, the dimension of $V \otimes W$ is the product of dimensions of V and W.

Hint: For V^*, to show linear independence, suppose some linear combination of the e_i^* is zero. By evaluating this linear combination at each basis vector e_i, you should be able to deduce that each coefficient is zero. To show these linear functionals span V^*, let $\phi : V \to \mathbb{R}$ be an arbitrary linear functional. Show that $\phi = \sum_i \phi(e_i)e_i^*$ by evaluating each side on an arbitrary basis vector e_j.

For the tensor product, I find it easiest to think abstractly. Write V as $\mathbb{R} \oplus \cdots \oplus \mathbb{R}$ where there are $\dim(V)$ copies (one for each basis vector); this is just the usual identification of V with $\mathbb{R}^{\dim(V)}$ that a basis gives you. Express W similarly. Now, for general vector spaces V_1, V_2, V_3, construct isomorphisms between $V_1 \otimes (V_2 \oplus V_3)$ and $(V_1 \otimes V_2) \oplus (V_1 \otimes V_3)$ (i.e. show that tensor product distributes over direct sum). This sort of thing is easiest to prove using universal properties: to define a linear map out of $V_1 \otimes (V_2 \oplus V_3)$, define a bilinear map out of $V_1 \times (V_2 \oplus V_3)$, etc. Once you have this, you can expand out $V \otimes W$ as $\mathbb{R} \oplus \cdots \oplus \mathbb{R}$ where there are $\dim(V)\dim(W)$ copies and show that under this identification, the standard basis of $\mathbb{R}^{\dim(V)\dim(W)}$ is identified with the set $\{e_i \otimes f_j\} \subset V \otimes W$.

(2) Now we study how tensor products interact with dual spaces. If V and W are vector spaces over $\mathbb{R}$, there is a map $T : V^* \times W^* \to (V \otimes W)^*$ sending (ϕ, ψ) to $(v \otimes w) \mapsto \phi(v)\psi(w)$. Prove that this map is well-defined and bilinear (and thus induces a linear map $V^* \otimes W^* \to (V \otimes W)^*$). If V and W are finite-dimensional, prove that this induced map is an isomorphism.

Hint: For well-definedness, check that given ϕ and ψ, the map from $V \times W$ to $\mathbb{R}$ given by $(v, w) \mapsto \phi(v)\psi(w)$ is bilinear. For bilinearity of T, check that $T(\phi_1 +$

$c\phi_2, \psi) = T(\phi_1, \psi) + cT(\phi_2, \psi)$. You can do this by showing that both sides agree when applied to an element $v \otimes w$ of $V \otimes W$. This shows linearity in the first slot; linearity in the second slot is similar.

Now, when V and W are finite-dimensional, let $\{e_i\}$ be a basis for V and let $\{f_j\}$ be a basis for W. Then $\{e_i^* \otimes f_j^*\}$ is a basis for $V^* \otimes W^*$ and $\{(e_i \otimes f_j)^*\}$ is a basis for $(V \otimes W)^*$. Prove that the map induced by T sends the first basis bijectively to the second basis.

Remark. The left side $(V \otimes W)^*$ of the above isomorphism can be identified with the space of bilinear maps from $V \times W$ to $\mathbb{R}$. This problem shows that this space of bilinear maps can also be described as $V^* \otimes W^*$, when V and W are finite-dimensional. More generally, if all spaces V_i are finite-dimensional we can identify $V_1^* \otimes \cdots V_n^*$ with the space of multilinear maps from $V_1 \times \cdots \times V_n$ into $\mathbb{R}$.

Even more generally, if all spaces V_i and W_j are finite-dimensional, we can identify $V_1^* \otimes \cdots \otimes V_n^* \otimes W_1 \otimes \cdots W_m$ with the space of multilinear maps from $V_1 \times \cdots \times V_n$ to $W_1 \otimes \cdots \otimes W_n$. A special case is when $n = m = 1$; we can identify $V^* \otimes W$ (or $W \otimes V^*$) with the space of linear maps from V to W. This identification is compatible with the identifications (linear transformations $\leftrightarrow$ matrices, given bases) and (matrices $\leftrightarrow$ certain tensors, given bases), as mentioned in last week's homework.

(3) At this point, we can define differential k-forms:

Definition. We let $\text{Alt}_k(\mathbb{R}^n, \mathbb{R})$ be the vector space of multilinear maps $\alpha : \mathbb{R}^n \times \cdots \mathbb{R}^n \to \mathbb{R}$ with

$$\alpha(v_1, \ldots, v_{i+1}, v_i, \ldots, v_n) = -\alpha(v_1, \ldots, v_i, v_{i+1}, \ldots, v_n)$$

for $1 \leq i < n$. A *differential k-form* on $U \subset \mathbb{R}^n$ is a function

$$\alpha : U \to \text{Alt}_k(\mathbb{R}^n, \mathbb{R}).$$

C^r regularity for α can be defined by looking at its coordinates in any basis for the finite-dimensional vector space $\text{Alt}_k(\mathbb{R}^n, \mathbb{R})$, and this is independent of the choice of basis.

By the above problems, $\text{Alt}_k(\mathbb{R}^n, \mathbb{R})$ can be viewed as a subspace of $(\mathbb{R}^n)^* \otimes \cdots (\mathbb{R}^n)^*$, so a differential k-form is a particular type of rank-k contravariant tensor field on U (namely one that satisfies the "alternating property" at each point). While this perspective is not strictly speaking necessary to define differential k-forms, it will prove very useful when defining wedge products.

We will focus first on 1-forms, which are a bit simpler because they don't require the tensor product.

Example. Let $U \subset \mathbb{R}^n$ be open. A *differential* 1*-form* on U is a function $\alpha : U \to (\mathbb{R}^n)^*$. Given any basis $\{\phi_1, \ldots, \phi_n\}$ for $(\mathbb{R}^n)^*$, we can write $\alpha(p) = a_1(p)\phi_1 + \cdots + a_n(p)\phi_n$ where $a_1, \ldots, a_n$ are functions from U to $\mathbb{R}$. We say that α is of class C^r $(1 \leq r \leq \infty)$ if all the functions a_i are of class C^r; since changes of basis on $\mathbb{R}^n$ are C^∞ (even linear) functions, this notion is independent of the choice of basis $\{\phi_1, \ldots, \phi_n\}$.

We are already familiar with differential 0-forms:

Example. Let $U \subset \mathbb{R}^n$ be open. A *differential* 0*-form* on U is a function $f : U \to \mathbb{R}$.

There are four fundamental operations on differential forms that we need to understand:

- Exterior derivative: if α is a k-form (say C^∞), then we have a $k+1$-form $d\alpha$ (this operation will generalize the gradient, curl, and divergence operations).
- Wedge product: if α is a k-form and β is an l-form, then we have a $k+l$-form $\alpha \wedge \beta$ (this operation will generalize the cross product of vectors in $\mathbb{R}^3$).
- Pullback: if α is a k-form on U and $F : V \to U$ is smooth where $V \subset \mathbb{R}^m$ is open (for some m), then we have a k-form $F^*(\alpha)$ on V.
- Integration: if α is an n-form on an n-dimensional cube $[0,1]^n \subset \mathbb{R}^n$ (it's okay that this isn't an open set although α should at least be continuous in the right sense), we have a real number
$$\int_{[0,1]^n} \alpha$$
(we can also integrate on more general rectangular sets than just $[0,1]^n$). Combined with pullbacks, this operation will let us generalize line integrals and surface integrals.

Before looking at k-forms for $k > 1$, we will study what we can for 0-forms and 1-forms. The wedge product of a zero-form f with any k-form is always defined as ordinary scalar multiplication by the value of f at each point; the pullback of a zero-form f by a function F is just defined to be $f \circ F$, and integration of zero-forms on $\mathbb{R}^0$ is just evaluation at the unique point $0 \in \mathbb{R}^0$. As we will see below, any one-form α on $[a,b]$ is $f(t)dt$ for some function f, and (e.g. assuming f is smooth or just Riemann integrable) we can just define $\int_{[a,b]} \alpha := \int_a^b f(t)dt$ as usual.

For the less trivial operations, we will start by studying the exterior derivative df of a zero-form f (also known as the differential of f), closely related to the gradient of f.

Definition. Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be a smooth function (0-form). From f, define a differential 1-form df by the equation
$$df(p) = (Df)_p$$
(in other words, df is just Df, which at a point p gives a linear map from $\mathbb{R}^n$ to $\mathbb{R}$).

Problem. Let $\{e_1, \ldots, e_n\}$ be the standard basis vectors of $\mathbb{R}^n$ and let $\{e_1^*, \ldots, e_n^*\}$ be their dual basis vectors. Let f be as above; prove that
$$df = \frac{\partial f}{\partial x^1} e_1^* + \cdots + \frac{\partial f}{\partial x^n} e_n^*$$
(we use "up indices" x^i rather than "down indices" x_i to match the physics conventions; a physicist may write df as $\frac{\partial f}{\partial x^i}$ or even as $\partial_i f$, which is a 1-form since the i index is down ("up indices in a denominator are down")).

Hint: It suffices to show that $df(p)(e_i) = \frac{\partial f}{\partial x^i}(p)$ for each i and each $p \in U$. Use the computation of $(Df)_p$ as a matrix from class, and multiply this matrix on the right by the standard basis vector e_i.

Remark. Given a basis for $\mathbb{R}^n$ (in particular, the standard basis), one can identify differential 1-forms on U with *vector fields* on U, i.e. functions from U to $\mathbb{R}^n$ (rather than $(\mathbb{R}^n)^*$). The vector field associated to the 1-form df is the *gradient* of f, denoted ∇f or $\mathrm{grad}(f)$:

$$\nabla f = \frac{\partial f}{\partial x^1} e_1 + \cdots + \frac{\partial f}{\partial x^n} e_n.$$

In general, df is a bit more natural of an object than ∇f; it doesn't require any choice of basis to define.

Importantly, one can also identify 1-forms and vector fields by picking an inner product for $\mathbb{R}^n$, rather than a basis. On a general smooth manifold M, the 1-form df for a function $f : M \to \mathbb{R}$ is always defined, but the gradient ∇f requires a choice of "Riemannian metric" on M (inner product on each tangent space). In Einstein notation, one sometimes writes "$\partial^i f$" for the gradient of f, as opposed to the 1-form $\partial_i f$. The Riemannian metric itself can be viewed as a function on M with values in $(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$, since an inner product on $\mathbb{R}^n$ is (in particular) a bilinear map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ (see below). Thus, as mentioned in last week's homework, the metric is determined (locally) by functions g_{ij} on M for $1 \leq i, j \leq \dim(M)$. The gradient of f is determined uniquely by the formula $\partial_i f = g_{ij}\partial^j f$ (note the implicit sum over j); for the usual metric on $\mathbb{R}^n$, $g_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

(4) Recall that for $1 \leq i \leq n$, we have a coordinate projection function $x^i : U \to \mathbb{R}$ (sending a point in U to its i^{th} coordinate) and thus a 1-form dx^i.

Problem. Prove that for all $p \in U$, we have $dx^i(p) = e_i^*$, where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$.

Hint: You can use the previous problem; just compute the partial derivative of the function x^i with respect to each x^j.

Remark. It follows that if $f : U \to \mathbb{R}$ is a smooth function as above, then we can write

$$df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n,$$

a natural-looking formula.

(5) Now we will define pullbacks of 1-forms:

Definition. Let $V \subset \mathbb{R}^m$ be open and let α be a differential 1-form on V. Let $U \subset \mathbb{R}^n$ be open and let $F : U \to V$ be a smooth function. The pullback $F^*(\alpha)$ is the differential 1-form on U defined at a point $p \in U$ by

$$(F^*(\alpha))_p(v) := \alpha_{F(p)}((DF)_p(v))$$

for $v \in \mathbb{R}^n$ (note that $(DF)_p(v) \in \mathbb{R}^m$ so it makes sense to evaluate $\alpha_{F(p)}$ on the vector $(DF)_p(v)$).

The idea is that to pull back a differential 1-form by F, you push forward the corresponding "input vector" v by DF; the same idea is used to define pullbacks of k-forms.

Problem. Prove the following facts about pullbacks of 1-forms:

(a) If $f : V \to \mathbb{R}$ is a smooth function and $F : U \to V$ is smooth, then

$$F * (df) = d(f \circ F)$$

(which equals $d(F^*(f))$, so that pullbacks commute with exterior derivatives acting on zero-forms).

(b) If $\alpha = f_1\alpha_1 + \cdots + f_N\alpha_N$, then

$$F^*(\alpha) = (f_1 \circ F)F^*(\alpha_1) + \cdots + (f_N \circ F)F^*(\alpha_N)$$

(c) If $\alpha = f_1 dx^1 + \cdots + f_m dx^m$ and $\vec{r} : [a, b] \to V$ is a smooth path with components $r_1, \ldots, r_m$, then

$$\vec{r}^*(\alpha) = f_1(\vec{r}(t))r_1'(t)dt + \cdots + f_m(\vec{r}(t))r_m'(t)dt,$$

where we use t for the coordinate in $[a, b]$.

Hint: For the first part, use that $df := Df$ and apply the chain rule to show that both sides act the same way on any vector v. For the second part, it also helps to apply both sides to a vector v, so you can use the definition of pullback to phrase things in terms of more familiar concepts. The third part should follow from the first two parts.