MATH 425b Homework 13

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Problem 1

Show that if V is finite-dimensional with basis $\{e_i\}$, then the linear functionals $\{e_i^*\}$ form a basis for V^*, where $e_i^*(e_j) = \delta_{i,j}$. In particular, V^* has the same dimension as V. Also show that if V, W are finite-dimensional with bases $\{e_i\}$ and $\{f_i\}$ then the elements $e_i \oplus f_j$ form a basis for $V \oplus W$. In particular, the dimension of $V \oplus W$ is the product of $\dim(V)$ and $\dim(W)$.

Proof. For convenience denote $\dim(V) = n$, i.e., $\{e_i\} = \{e_i\}_{i=1}^n$. To show the linear independence of $\{e_i^*\}$, suppose $\sum_{i=1}^n c_i e_i^* = 0$ (the zero functional). Evaluating this linear combination at e_j we see

$$\sum_{i=1}^n c_i e_i^*(e_j) = \sum_{i=1}^n c_i \delta_{i,j} = c_i = 0$$

and so all coefficients c_i's must be 0. Thus $\{e_i^*\}$ is linearly independent. Now we show that it spans V^*. Indeed, for $\varphi \in V^*$, we have

$$\sum_{i=1}^n \varphi(e_i) e_i^*(e_j) = \varphi(e_j)$$

for all $e_j \in \{e_i\}$. Since φ is linear, $\varphi(v) = \sum_{i=1}^n \varphi(e_i) e_i^*(v)$ for all $v \in V$. Indeed $\{e_i^*\}$ forms a basis for V^*.

Per the hint, we first show that direct sum can be distributed over tensor products, i.e.,

$$V_1 \otimes (V_2 \oplus V_3) \cong (V_1 \otimes V_2) \oplus (V_1 \otimes V_3).$$

Per the hint, consider a linear transformation $f : V_1 \times (V_2 \oplus V_3) \to (V_1 \otimes V_2) \oplus (V_1 \otimes V_3)$ by

$$(v_1, (v_2, v_3)) \mapsto (v_1 \otimes v_2, v_1 \otimes v_3) \text{ for } v_1 \in V_1, v_2 \in V_2, v_3 \in V_3.$$

Indeed this is a bilinear mapping (we have shown in HW12 that the mapping $(v, w) \mapsto v \otimes w$ is bilinear). Therefore by the universal property, there exists a linear $g : V_1 \otimes (V_2 \oplus V_3) \to (V_1 \otimes V_2) \oplus (V_1 \otimes V_3)$.

$$\begin{array}{ccc} V_1 \times (V_2 \oplus V_3) & & \\ \downarrow \pi & \searrow^{f} & \\ V_1 \otimes (V_2 \oplus V_3) & \xrightarrow[g]{} & (V_1 \otimes V_2) \oplus (V_1 \otimes V_3) \end{array}$$

Therefore, if we identify V as $\mathbb{R} \underbrace{\oplus \cdots \oplus}_{\dim(V) \text{ times}} \mathbb{R}$ and W as $\mathbb{R} \underbrace{\oplus \cdots \oplus}_{\dim(W) \text{ times}} \mathbb{R}$, applying the distributivity above tells us that we can identify $V \otimes W$ with $(\mathbb{R} \otimes \mathbb{R}) \underbrace{\oplus \cdots \oplus}_{\dim(V)\dim(W) \text{ times}} (\mathbb{R} \otimes \mathbb{R})$. Since $\mathbb{R} \otimes \mathbb{R} \cong \mathbb{R}$, this can be further identified with the direct sum of $\dim(V)\dim(W)$ copies of $\mathbb{R}$. Naturally we have the standard basis for $\mathbb{R}^{\dim(V)\dim(W)}$, and this can be idenfitied with $\{e_i \otimes f_j\}$ via the (composite) map $e_i \otimes f_j \mapsto (e_i, f_j) \mapsto \delta_{i,j}$. This proves the claim. □

Problem 2

If V and W are vector spaces over $\mathbb{R}$, there is a map $T : V^* \times W^* \to (V \otimes W)^*$ sending (φ, ψ) to (the functional) $(v \otimes w) \mapsto \varphi(v)\psi(w)$. Prove that this map is well-defined and bilinear (and thus induces a linear map $V^* \otimes W^* \to (V \otimes W^*)$). If V and W are finite-dimensional, prove that this induced map is an isomorphism.

Proof. Since φ and ψ are linear functionals, the mapping $(v, w) \mapsto \varphi(v)\psi(w)$ is bilinear. Therefore, once φ, ψ are given, the mapping $(v \otimes w) \mapsto \varphi(v)\psi(w)$ is uniquely determined. Hence T is well-defined.
Now we show the bilinearity of T (not of its output). Consider $T(\varphi_1 + c\varphi_2, \psi)$. The output is the linear functional that maps $(v \oplus w)$ to $(\varphi_1 + c\varphi_2)(v)\psi(w)$. By definition this is equal to $\varphi_1(v)\psi(w) + c\varphi_2(v)\psi(w)$, which is equal to $T(\varphi_1, \psi) + cT(\varphi_2, \psi)$. Hence T is linear with respect to the first argument, and one can show analogously that it is also linear w.r.t. the second. Bilinearity then follows.

Now we assume $\dim(V) = n < \infty$ and $\dim(W) = m < \infty$. Let $\{e_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^m$ be their bases, respectively. For convenience call this induced map J. To show that J defines an isomorphism, it suffices to check that J restricted to $\{e_i^* \oplus f_j^*\}$ is a bijection to $\{(e_i \oplus f_j)^*\}$ (the rest of the claim, i.e., bijectivity between $V^* \times W^*$ and $(V \oplus W)^*$, follows by bilinearity and properties of basis). For injectivity, if $e_i^*, \tilde{e}_i^*, f_j^*, f_j^*$ satisfy

$$S(e_i^* \otimes f_j^*) = S(\tilde{e}_i^* \otimes \tilde{f}_j^*) \implies e_i^*(v) f_j^*(w) = \tilde{e}_i^*(v) \tilde{f}_j^*(w)$$

for all $v \otimes w \in V \otimes W$, then fixing w tells us $e_i^*(v) = \tilde{e}_i^*(v)$ for all v, i.e., $\tilde{e}_i^* = e_i^*$ and likewise $\tilde{f}_j^* = f_j^*$. Hence S is injective. For surjectivity, simply notice that the pre-image of the functional $(v \otimes w) \mapsto \tilde{e}(v)\tilde{f}(w)$ is simply $(\tilde{e} \otimes \tilde{f}) \in V^* \otimes W^*$. □

Problem 3

Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be a smooth function (0-form). From f, define a differential 1-form $\mathrm{d}f$ by $\mathrm{d}f(p) = (\mathrm{D}f)_p$. Let $\{e_1, \ldots, e_n\}$ be the standard basis vectors of $\mathbb{R}^n$ and let $\{e_1^*, \ldots, e_n^*\}$ be their dual basis vectors. Prove that

$$\mathrm{d}f = \frac{\partial f}{\partial x^1} e_1^* + \ldots + \frac{\partial f}{\partial x^n} e_n^*.$$

Proof. By definition $(\mathrm{D}f)_p$ is given by

$$\left[\frac{\partial f}{\partial x^1}(p) \quad \cdots \quad \frac{\partial f}{\partial x^n}(p) \right].$$

Notice that, for any e_i (standard basis vector), since it would only keep $\partial f / \partial x^i$ under multiplication, we have

$$\mathrm{d}f(p)(e_i) = (\mathrm{D}f)_p(e_i) = \frac{\partial f}{\partial x^i}(p) \quad \text{for all } p \in U.$$

Therefore, if $x = \sum_{i=1}^n c_i e_i$, we have

$$\mathrm{d}f(p)(x) = (\mathrm{D}f)_p(x) = \sum_{i=1}^n c_i (\mathrm{D}f)_p(e_i) = \sum_{i=1}^n c_i \frac{\partial f}{\partial x^i}(p)$$

so indeed

$$\mathrm{d}f = \frac{\partial f}{\partial x^1} e_1^* + \ldots + \frac{\partial f}{\partial x^n} e_n^*.$$

□

Problem 4

Recall that for $1 \leqslant i \leqslant n$ we have a coordinate projection function $x^i : U \to \mathbb{R}$ sending a point in U to its i^{th} coordinate and thus a 1-form $\mathrm{d}x^i$. Prove that for all $p \in U$, we have $\mathrm{d}x^i(p) = e_i^*$.

Proof. Using the previous part, $\mathrm{d}x^i = \frac{\partial x^i}{\partial x^1} e_1^* + \ldots + \frac{\partial x^i}{\partial x^n} e_n^* = \frac{\partial x^i}{\partial x^i} e_i^* = e_i^*$ so $\mathrm{d}x^i(p) = e_i^*$ for all p. □

Problem 5

Prove the following facts about pulls of 1-forms:

(1) If $f : V \to \mathbb{R}$ is a smooth function and $F : U \to V$ is smooth then

$$F^*(\mathrm{d}f) = \mathrm{d}(f \circ F).$$

(2) If $\alpha = f_1\alpha_1 + \ldots + f_N\alpha_N$ then

$$F^*(\alpha) = (f_1 \circ F)F^*(\alpha_1) + \ldots + (f_N \circ F)F^*(\alpha_N).$$

(3) If $\alpha = f_1\mathrm{d}x^1 + \ldots + f_m\mathrm{d}x^m$ and $r : [a, b] \to V$ is a smooth path with component $r_1, \ldots, r_m$, then

$$r^*(\alpha) = f_1(r(t))r_1'(t)\mathrm{d}t + \ldots + f_m(r(t))r_m'(t)\mathrm{d}t.$$

Proof. (1) Since $\mathrm{d}f = \mathrm{D}f$, we have

$$\begin{aligned} F^*(\mathrm{d}f)_p(v) &= F^*(\mathrm{D}f)_p(v) = \mathrm{D}f_{F(p)}(\mathrm{D}F_p(v)) \\ &= \mathrm{D}(f \circ F)_p(v) = \mathrm{d}(f \circ F)(p)(v). \end{aligned}$$

(2) Applying definition, we have

$$\begin{aligned} (F^*(\alpha))_p(v) &= \alpha_{F(p)}((\mathrm{D}F)_p(v)) \\ &= (f_1\alpha_1 + \ldots + f_N\alpha_N)_{F(p)}((\mathrm{D}F)_p(v)) \\ &= (f_1\alpha_1)_{F(p)}((\mathrm{D}F)_p(v)) + \ldots + (f_N\alpha_N)_{F(p)}((\mathrm{D}F)_p(v)) \\ &= (f_1 \circ F)(v)F^*(\alpha_1)(v) + \ldots + (f_N \circ F)(v)F^*(\alpha_n)(v). \end{aligned}$$

(3) Using (1) and (2),

$$\begin{aligned} r^*(\alpha) &= r^*(f_1\mathrm{d}x^1 + \ldots + f_m\mathrm{d}x^m) \\ &= f_1(r(t))r^*(\mathrm{d}x^1) + \ldots + f_m(r(t))r^*(\mathrm{d}x^m) \\ &= f_1(r(t))r_1'(t)\mathrm{d}t + \ldots + f_m(r(t))r_m'(t)\mathrm{d}t. \end{aligned}$$

□