HOMEWORK, WEEK 14

This assignment is due Monday, April 26. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

Along with gradients, there are related differential operators in multivariable calculus, namely the curl and divergence of a vector field F in three dimensions. The theorems $\mathrm{curl} \circ \mathrm{grad} = 0$ and $\mathrm{div} \circ \mathrm{curl} = 0$ are often discussed, and related concepts are introduced. For example, a *conservative vector field* is a vector field F with $F = \mathrm{grad}(f)$ for some function f. If $\mathrm{curl}(F) = 0$, one often asks whether F is a conservative vector field; the answer to this question depends on the topology of $U \subset \mathbb{R}^3$.

The most natural setting for these concepts is that of differential k-forms for $k \geq 0$, where these ideas make sense in n dimensions and become considerably more unified. In particular, we will begin working with k-forms for $k > 1$. An important goal will be understanding the standard identifications

- 0-forms on $\mathbb{R}^3 \leftrightarrow$ functions on $\mathbb{R}^3$
- 1-forms on $\mathbb{R}^3 \leftrightarrow$ vector fields on $\mathbb{R}^3$
- 2-forms on $\mathbb{R}^3 \leftrightarrow$ vector fields on $\mathbb{R}^3$
- 3-forms on $\mathbb{R}^3 \leftrightarrow$ functions on $\mathbb{R}^3$;

with all k-forms on $\mathbb{R}^3$ being zero if $k > 3$.

Remark. Implicitly, these identifications use the standard inner product / Riemannian metric on $\mathbb{R}^3$, as with the identification between df and ∇f mentioned in last week's homework. Note that when working only with functions and vector fields, it may not be immediately clear whether a vector field $\vec{F}$ came from a 1-form or a 2-form (or a legitimate vector field); similarly, a given function f might represent a 3-form instead of a 0-form.

Given these identifications, we will see that:

- The gradient operation grad (or ∇), taking functions to vector fields, becomes identified with the exterior derivative d taking 0-forms to 1-forms.
- The curl operation curl (or $\nabla\times$), taking vector fields to vector fields, becomes identified with the exterior derivative d taking 1-forms to 2-forms.
- The divergence operation div (or $\nabla\cdot$), taking vector fields to functions, becomes identified with the exterior derivative d taking 2-forms to 3-forms.

The relations $\mathrm{curl} \circ \mathrm{grad} = 0$ and $\mathrm{div} \circ \mathrm{curl} = 0$ are special cases of the fundamental property $d^2 = 0$ for the exterior derivative.

(1) We start with the necessary preliminaries for identifying differential forms on $\mathbb{R}^3$ with functions and vector fields as above. Recall the definition of differential k-forms from last week's homework:

Definition. If $U \subset \mathbb{R}^n$ is open, a differential k-form on U is a function α from U to $(\mathbb{R}^n)^* \otimes \cdots \otimes (\mathbb{R}^n)^*$ (with k factors in the tensor product) which is "alternating" (or anti-symmetric) in the following sense: viewing $\alpha(p) \in (\mathbb{R}^n)^* \otimes \cdots \otimes (\mathbb{R}^n)^*$ as an element of $(\mathbb{R}^n \otimes \cdots \mathbb{R}^n)^*$ by last week's homework (i.e. a multilinear map from $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ to $\mathbb{R}$), the map $\alpha(p)$ is *alternating* in the sense that for all k-tuples $(v_1, \cdots, v_k)$ of vectors in $\mathbb{R}^n$, we have

$$\alpha(p)(v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_k) = -\alpha(p)(v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k)$$

for any i with $1 \leq i \leq k-1$. We say that α is of class C^r if its coordinates in the standard basis of $(\mathbb{R}^n)^* \otimes \cdots \otimes (\mathbb{R}^n)^*$ (equivalently, any basis for this vector space) are C^r functions from U to $\mathbb{R}$.

In the convenient shorthand of last week's homework, a differential k-form on U is a function from $U \subset \mathbb{R}^n$ to $\mathrm{Alt}_k(\mathbb{R}^n, \mathbb{R})$. For example, a differential 2-form α, evaluated at a point $p \in U$, gives a bilinear map $\alpha(p)$ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ such that $\alpha(p)(v, w) = -\alpha(p)(w, v)$ for all $v, w \in \mathbb{R}^n$.

The identifications we want will become more plausible once we know that

- $\dim \mathrm{Alt}_0(\mathbb{R}^3, \mathbb{R}) = 1$
- $\dim \mathrm{Alt}_1(\mathbb{R}^3, \mathbb{R}) = 3$
- $\dim \mathrm{Alt}_2(\mathbb{R}^3, \mathbb{R}) = 3$
- $\dim \mathrm{Alt}_3(\mathbb{R}^3, \mathbb{R}) = 1$;

in general, $\dim \mathrm{Alt}_k(\mathbb{R}^n, \mathbb{R}) = \binom{n}{k}$ (and equals zero unless $0 \leq k \leq n$). The case of Alt_0 is tautological since the tensor product of zero copies of $\mathbb{R}^n$ is defined to be $\mathbb{R}$. The case of Alt_1 just says that $(\mathbb{R}^n)^*$ has dimension n, which follows from last week's homework.

These dimension counts (and the identifications we want) will follow from the fact that the wedge products

$$\{dx^{i_1} \wedge \cdots \wedge dx^{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

form a basis for $\mathrm{Alt}^k(\mathbb{R}^n, \mathbb{R})$, but for this to make sense, we first need to define wedge products.

Remark. In my view, the tensor product abstraction really shines when defining wedge products. One constructs isomorphisms between $\mathrm{Alt}^k(\mathbb{R}^n, \mathbb{R})$ and the summand $\wedge^k(\mathbb{R}^n)^*$ of the "exterior algebra" $\wedge^\cdot(\mathbb{R}^n)^*$, on which multiplication has a very natural definition. Wedge products of elements of $\mathrm{Alt}^k(\mathbb{R}^n, \mathbb{R})$ are most naturally defined using the isomorphisms to $\wedge^k(\mathbb{R}^n)^*$. We will discuss this approach a bit in lecture; for now, we will build intuition by taking a more concrete (but equivalent) approach to wedge products.

Definition. If $\phi, \psi \in (\mathbb{R}^n)^*$, define an alternating bilinear map $\phi \wedge \psi$ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ by

$$(\phi \wedge \psi)(v, w) := (\phi(v)\psi(w) - \phi(w)\psi(v)).$$

If α and β are 1-forms on U, then $\alpha \wedge \beta$ is a 2-form on U defined by $(\alpha \wedge \beta)(p) := \alpha(p) \wedge \beta(p)$ (wedge products of differential forms are defined "pointwise").

Note that we have $\psi \wedge \phi = -\phi \wedge \psi$ (so $\phi \wedge \phi = 0$), and that wedge products are bilinear:

$$(c_1\phi_1 + c_2\phi_2) \wedge \psi = c_1\phi_1 \wedge \psi + c_2\phi_2 \wedge \psi$$

and similarly in the second slot.

Problem. Show that

$$\{dx \wedge dy, dx \wedge dz, dy \wedge dz\}$$

is a basis for the vector space of alternating bilinear maps from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$. (Strictly speaking, dx, dy, and dz are differential 1-forms defined on $\mathbb{R}^3$, so we should evaluate them at a point p in $\mathbb{R}^3$, but their values are independent of p and it is standard to just write dx, dy, and dz).

Hint: By last week's homework, you have $dx(p) = e_1^*$, $dy(p) = e_2^*$, and $dz(p) = e_3^*$ for any $p \in \mathbb{R}^3$, so you want to show that $\{e_1^* \wedge e_2^*, e_1^* \wedge e_3^*, e_2^* \wedge e_3^*\}$ is a basis for this space of alternating bilinear maps.

You know from the definition that each $e_i^* \wedge e_j^*$ is bilinear and alternating (you don't need to prove this). You want to show that this set of three maps is linearly independent and spans the space of alternating bilinear maps. For independence, consider a general linear combination and evaluate it at pairs (e_i, e_j) of standard basis vectors for $i < j$. You should get that each coefficient is zero.

To show that $\{e_1^* \wedge e_2^*, e_1^* \wedge e_3^*, e_2^* \wedge e_3^*\}$ spans the space of alternating bilinear maps, consider an arbitrary map $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that is alternating and bilinear. By evaluating Φ on pairs (e_i, e_j) of standard basis vectors for $i < j$, you will get three real numbers. Try to use these numbers to construct a linear combination of $e_1^* \wedge e_2^*$, $e_1^* \wedge e_3^*$, and $e_2^* \wedge e_3^*$ such that this linear combination agrees with Φ on all pairs of standard basis vectors (e_i, e_j) (not necessarily with $i < j$).

(2) In multivariable calculus, one often studies the cross product of vectors in $\mathbb{R}^3$, which is another vector in $\mathbb{R}^3$. Viewed in terms of the dual space $(\mathbb{R}^3)^*$ instead of $\mathbb{R}^3$, you will show in this problem that the cross product becomes a special case of the wedge product, taking in two elements of $(\mathbb{R}^3)^* = \mathrm{Alt}_1(\mathbb{R}^3, \mathbb{R})$ and producing an element of the three-dimensional space $\mathrm{Alt}_2(\mathbb{R}^3, \mathbb{R})$.

For an element $\alpha = a_1 e_1^* + a_2 e_2^* + a_3 e_3^*$ of $\mathbb{R}_3^* = \mathrm{Alt}_1(\mathbb{R}^3, \mathbb{R})$, define $\Phi(\alpha) \in \mathbb{R}^3$ to be the vector (a_1, a_2, a_3), or equivalently $a_1 e_1 + a_2 e_2 + a_3 e_3$. Similarly, for an element

$$\alpha = a_{12} e_1^* \wedge e_2^* + a_{13} e_1^* \wedge e_3^* + a_{23} e_2^* \wedge e_3^*$$

of $\mathrm{Alt}_2(\mathbb{R}^3, \mathbb{R})$, define $\Phi(\alpha) \in \mathbb{R}^3$ to be the vector $(a_{23}, -a_{13}, a_{12})$, or equivalently $a_{23} e_1 - a_{13} e_2 + a_{12} e_3$.

Remark. Seeing that this is the natural vector in $\mathbb{R}^3$ to define given an element of $\mathrm{Alt}_2(\mathbb{R}^3, \mathbb{R})$ involves looking at the Hodge star operator, which will make an appearance in next week's problem set. The minus sign disappears if we use the basis vector $e_3^* \wedge e_1^*$ rather than $e_1^* \wedge e_3^*$ in the basis for $\mathrm{Alt}_2(\mathbb{R}^3, \mathbb{R})$.

Problem. Show that for $\alpha, \beta \in (\mathbb{R}^3)^*$, we have

$$\Phi(\alpha \wedge \beta) = \Phi(\alpha) \times \Phi(\beta),$$

where $\times$ denotes the usual cross product of vectors in $\mathbb{R}^3$.

Hint: Write out $\alpha \wedge \beta$ in terms of basis expansions $\alpha = a_1 e_1^* + a_2 e_2^* + a_3 e_3^*$ and $\beta = b_1 e_1^* + b_2 e_2^* + b_3 e_3^*$. Then expand this out using the basic properties of wedge products stated above, write the result in terms of the basis for $\mathrm{Alt}_2(\mathbb{R}^3, \mathbb{R})$ that we're using, and compute $\Phi(\alpha \wedge \beta)$ with these basis coefficients. Compare the result to $\Phi(\alpha) \times \Phi(\beta)$, computed as in multivariable calculus.

(3) More generally, the abstract definition of wedge products will imply that if $\phi_1, \ldots, \phi_k$ are in $(\mathbb{R}^n)^*$, then

$$\phi_1 \wedge \cdots \wedge \phi_k = \sum_{\sigma \in S_k} (-1)^{\mathrm{sign}(\sigma)} \phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(k)},$$

where S_k is the symmetric group on k letters (of size $k!$) and $\mathrm{sign}(\sigma)$ is the number of transpositions modulo 2 in any factorization of σ into transpositions (if these ideas are unfamiliar, don't worry; you won't be asked to prove things about symmetric groups or permutations). The abstract definition also implies basic properties like associativity of wedge products, and makes it clear that wedge products of elements of $(\mathbb{R}^n)^* = \mathrm{Alt}_1(\mathbb{R}^n, \mathbb{R})$ form a spanning set for $\mathrm{Alt}_k(\mathbb{R}^n, \mathbb{R})$. Thus, the above formula suffices for computing all wedge products.

In particular, for $\phi_1, \phi_2, \phi_3 \in (\mathbb{R}^3)^*$, we have

$$\begin{aligned} \phi_1 \wedge \phi_2 \wedge \phi_3 = {} & \phi_1 \otimes \phi_2 \otimes \phi_3 \\ & - \phi_1 \otimes \phi_3 \otimes \phi_2 \\ & + \phi_2 \otimes \phi_3 \otimes \phi_1 \\ & - \phi_2 \otimes \phi_1 \otimes \phi_3 \\ & + \phi_3 \otimes \phi_1 \otimes \phi_2 \\ & - \phi_3 \otimes \phi_2 \otimes \phi_1, \end{aligned}$$

which we can take as a definition if we want.

Problem. Show, using the above concrete definition, that

$$\{dx \wedge dy \wedge dz\}$$

is a basis for $\mathrm{Alt}_3(\mathbb{R}^3, \mathbb{R})$.

Hint: You should show that $dx \wedge dy \wedge dz$ is alternating, since it's not as immediate as before, but since this is a bit tedious you can just show one representative computation (things are cleaner if you use the general formula above plus facts about permutations, but I don't want to assume knowledge of these facts). So, for example, you can just show that $\phi_1 \wedge \phi_2 \wedge \phi_3$ sends (v_2, v_1, v_3) to the negative of where it sends (v_1, v_2, v_3) (the "$i = 1$ case" in the definition we've been using for $\mathrm{Alt}_k(\mathbb{R}^n, \mathbb{R})$). You can just say the "$i = 2$ case" is similar.

You can check $dx \wedge dy \wedge dz$ is nonzero by plugging in the appropriate (v_1, v_2, v_3), so to finish the proof, it suffices to show that $\mathrm{Alt}_3(\mathbb{R}^3, \mathbb{R})$ is one-dimensional. Try to do this cleanly by applying a standard linear-algebra fact about "uniqueness of determinants" (if this fact is unfamiliar, you can ask me).

(4) We have the following consequences of the above problems plus what we've said before:

- A zero-form on $\mathbb{R}^3$ is a function f on $\mathbb{R}^3$.
- A one-form on $\mathbb{R}^3$ can be written uniquely as $f dx + g dy + h dz$ where f, g, h are functions on $\mathbb{R}^3$.
- A two-form on $\mathbb{R}^3$ can be written uniquely as $f dx \wedge dy + g dx \wedge dz + h dy \wedge dz$ where f, g, h are functions on $\mathbb{R}^3$.
- A three-form on $\mathbb{R}^3$ can be written uniquely as $f dx \wedge dy \wedge dz$ where f is a function on $\mathbb{R}^3$.

In general, a differential k-form α on $\mathbb{R}^n$ can be written uniquely as

$$\alpha = \sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

(we will see this when we discuss wedge products abstractly in lecture). This expression is especially convenient when defining the exterior derivative:

Definition. If

$$\alpha = \sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

is a differential k-form on an open subset U of $\mathbb{R}^n$, its exterior derivative $d\alpha$ is the $(k+1)$-form on U defined by

$$d\alpha = \sum_{1 \le i_1 < \cdots < i_k \le n} (df_{i_1,\ldots,i_k}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

(thus, if $\alpha = f$ is a zero-form, then $d\alpha = df$ as defined on last week's homework).

A crucial fact about d is that $d \circ d = 0$, i.e. $d(d(\alpha)) = 0$ for any k-form α. One can prove this in general; here we'll just prove it for forms on $\mathbb{R}^3$.

Problem. Prove that $d \circ d = 0$ for d acting on:
(a) 0-forms on $\mathbb{R}^3$
(b) 1-forms on $\mathbb{R}^3$
(the equation is automatic for 2-forms and 3-forms on $\mathbb{R}^3$). *You may assume Corollary 5.17 of Pugh, on equality of mixed partials, without proof.*

Hint: For 0-forms, first write $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$ as in the previous problem set. Then use the above definition to compute $d(df)$. Express the result in terms of the three basic 2-forms $dx \wedge dy$, $dx \wedge dz$, and $dy \wedge dz$. If you find something like $dy \wedge dx$, write it as $-dx \wedge dy$. If you find $dx \wedge dx$ (etc.), it's zero. The computation for 1-forms is similar, express the result of $d(d(\alpha))$ in terms of the basic 3-form $dx \wedge dy \wedge dz$ on $\mathbb{R}^3$.

Remark. Differential forms α with $d(\alpha) = 0$ are called *closed*; differential forms of the form $\alpha = d(\beta)$ are called *exact*. The equation $d \circ d = 0$ (or "$d^2 = 0$") says that all exact forms are closed. The equation $d^2 = 0$ is the key indicator of a "chain complex" of which one can take (co)homology groups $\frac{\ker(d)}{\mathrm{im}(d)}$. In this case (closed k-forms modulo exact k-forms) the result is called the k^{th} "de Rham cohomology" of the domain U. It is isomorphic to the k^{th} (real) "ordinary cohomology," which measures topological information about U (roughly "the number of $k+1$-dimensional holes"). The exterior derivative d also satisfies a Leibniz rule: we have

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

if α is a k-form.

(5) We can interpret k-forms on $\mathbb{R}^3$ in terms of familiar vector calculus quantities as follows, compatibly with the above problem on cross products

- A 0-form on $\mathbb{R}^3$ is just a function on $\mathbb{R}^3$ as usual.
- A 1-form $\alpha_x dx + \alpha_y dy + \alpha_z dz$ on $\mathbb{R}^3$ can be identified with the vector field $\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$.
- A 2-form $\alpha_x dy \wedge dz + \alpha_y dz \wedge dx + \alpha_z dx \wedge dy$ (note the choice of basis!) can also be identified with the vector field $\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$.
- A 3-form $f dx \wedge dy \wedge dz$ on $\mathbb{R}^3$ can be identified with the function f.

Note that a vector field on $\mathbb{R}^3$ could potentially represent a 1-form or a 2-form, and a function on $\mathbb{R}^3$ could potentially represent a 0-form or a 3-form.

Remark. The symmetry between k-forms and $3-k$-forms is related to *Poincaré duality*, an important relationship in algebraic topology.

Problem. Let α be a 1-form on $\mathbb{R}^3$, and let V be the corresponding vector field. Show that the 2-form $d\alpha$ corresponds to the vector field $\text{curl}(V)$.

Similarly, let β be a 2-form on $\mathbb{R}^3$, and let W be the corresponding vector field. Show that the 3-form $d\beta$ corresponds to the function $\text{div}(W)$.

Hint: For the first part, write α in terms of dx, dy, and dz so you can take $d\alpha$. Rewrite the result in the basis $\{dy \wedge dz, dz \wedge dx, dx \wedge dy\}$ so you can extract the corresponding vector field. The second part is similar.

Remark. From this problem, we recover the familiar identities $\text{curl} \circ \text{grad} = 0$ and $\text{div} \circ \text{curl} = 0$ from the equation $d^2 = 0$. The de Rham cohomology in degree 1 measures the maximal number of irrotational ($\text{curl} = 0$) vector fields on U that are "independent modulo vector fields that are gradients;" it's an interesting fact that this quantity depends only on the topology of U.

In fact, if U is homeomorphic to an open ball, all cohomology groups of U vanish for $k > 0$. On such $U \subset \mathbb{R}^3$, all vector fields F with $\text{curl}(F) = 0$ are conservative. However, if U has more complicated topology (e.g. if U is an open neighborhood of a circle or torus), then this property may fail.