MATH 425b Homework 13

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Definition

If $\varphi, \psi \in (\mathbb{R}^n)^*$, define an alternating bilinear map, the **wedge product**, $\varphi \wedge \psi$ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$, by

$$(\varphi \wedge \psi)(v, w) := \varphi(v)\psi(w) - \varphi(w)\psi(v).$$

Problem 1

Show that

$$\{\mathrm{d}x \wedge \mathrm{d}y, \mathrm{d}x \wedge \mathrm{d}z, \mathrm{d}y \wedge \mathrm{d}z\}$$

is a basis for the vector space of alternating bilinear maps from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$. *Strictly speaking,* $\mathrm{d}x, \mathrm{d}y$*, and* $\mathrm{d}z$ *are differential* 1*-forms defined on* $\mathbb{R}^3$ *so we should evaluate them at a point* $p \in \mathbb{R}^3$*, but their values are independent of* p *and it is standard to just write* $\mathrm{d}x, \mathrm{d}y$*, and* $\mathrm{d}z$.

Proof. From HW12 we know that $\mathrm{d}x(p) = e_1^*, \mathrm{d}y(p) = e_2^*$, and $\mathrm{d}z(p) = e_3^*$, regardless of choice of p. Thus it suffices to show that

$$\{e_1^* \wedge e_2^*, e_1^* \wedge e_3^*, e_2^* \wedge e_3^*\}$$

forms a basis. By definition these three are indeed bilinear and alternating, so it remains to check their linear independence and span. For linear independence, suppose for some $i < j$ we have

$$c_1(e_1^* \wedge e_2^*)(e_i, e_j) + c_2(e_1^* \wedge e_3^*)(e_i, e_j) + c_3(e_2^* \wedge e_3^*)(e_i, e_j) = 0.$$

Then,

$$\begin{aligned} 0 = \quad & c_1(e_1^*(e_i)e_2^*(e_j) - e_1^*(e_j)e_2^*(e_i)) \\ & + c_2(e_1^*(e_i)e_3^*(e_j) - e_1^*(e_i)e_3^*(e_j)) \\ & + c_3(e_2^*(e_i)e_3^*(e_j) - e_2^*(e_i)e_3^*(e_j)) \\ = {} & c_1(\delta_{1,i}\delta_{2,j} - \delta_{1,j}\delta_{2,i}) + c_2(\delta_{1,i}\delta_{3,j} - \delta_{1,j}\delta_{3,i}) + c_3(\delta_{2,i}\delta_{3,j} - \delta_{2,j}\delta_{3,i}) \\ = {} & c_1 \text{ if } (i,j) = (1,2), \quad c_2 \text{ if } (i,j) = (2,3), \quad \text{ and } c_3 \text{ if } (i,j) = (2,3). \end{aligned}$$

Therefore, letting i, j vary, we see $c_1 = c_2 = c_3 = 0$, i.e., these three wedge products are linearly independent. To show they span the entire space of alternating bilinear maps, let $\Phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be an arbitrary map with such property. We aim to find c_1, c_2, c_3 such that $\Phi = c_1(e_1^* \wedge e_2^*) + c_2(e_1^* \wedge e_3^*) + c_3(e_2^* \wedge e_3^*)$. In particular, adopting the Kronecker delta notation from above,

$$\Phi(e_1, e_2) = c_1 \qquad \Phi(e_1, e_3) = c_2 \qquad \text{and } \Phi(e_2, e_3) = c_3.$$

Therefore

$$\Phi = \Phi(e_1, e_2)(e_1^* \wedge e_2^*) + \Phi(e_1, e_3)(e_1^* \wedge e_3^*) + \Phi(e_2, e_3)(e_2^* \wedge e_3^*).$$

The alternating property of Φ guarantees that it agrees with our hypothesized linear combination on all $(e_i, e_j) \in \mathbb{R}^3 \times \mathbb{R}^3$, not just those with $i < j$. For $i > j$, take for example $(i, j) = (2, 1)$:

$$\Phi(e_2, e_1) = c_1(\delta_{1,2}\delta_{2,1} - \delta_{1,1}\delta_{2,2}) = -c_1.$$

For $i = j$, since Φ is alternating, we expect $\Phi(e_j, e_i) = \Phi(e_i, e_j) = -\Phi(e_j, e_i)$ so $\Phi(e_i, e_j) = 0$. Indeed,

$$c_1(e_{1,k}e_{2,k} - e_{1,k}e_{2,k}) + \ldots = 0.$$

Therefore $\{e_1^* \wedge e_2^*, e_1^* \wedge e_3^*, e_2^* \wedge e_3^*\}$ spans $\mathrm{Alt}_2(\mathbb{R}^3, \mathbb{R})$. □

Problem 2

For an element $\alpha = a_1 e_1^* = a_2 e_2^* + a_3 e_3^*$ of $\mathbb{R}^3 = \mathrm{Alt}_1(\mathbb{R}^3, \mathbb{R})$, define $\Phi(\alpha) \in \mathbb{R}^3$ to be the vector (a_1, a_2, a_3) or equivalently $a_1 e_1 + a_2 e_2 + a_3 e_3$. Similarly, for an element

$$\alpha = a_{12} e_1^* \wedge e_2^* + a_{13} e_1^* \wedge e_3^* + a_{23} e_2^* \wedge e_3^*$$

of $\mathrm{Alt}_2(\mathbb{R}^3, \mathbb{R})$, define $\Phi(\alpha) \in \mathbb{R}^3$ to be the vector $(a_{23}, -a_{13}, a_{12})$ or equivalently $a_{23} e_1 - a_{13} e_2 + a_{12} e_3$. (The minus sign would disappear if we used $e_3^* \wedge e_1^*$ rather than $e_1^* \wedge e_3^*$.)
Show that, for $\alpha, \beta \in (\mathbb{R}^3)^*$ we have

$$\Phi(\alpha \wedge \beta) = \Phi(\alpha) \times \Phi(\beta)$$

where $\times$ denotes the usual cross product of vectors in $\mathbb{R}^3$.

Proof. By definition of wedge products,

$$\begin{aligned}
\alpha \wedge \beta &= (a_1 e_1^* + a_2 e_2^* + a_3 e_3^*) \wedge (b_1 e_1^* + b_2 e_2^* + b_3 e_3^*) \\
&= \quad a_1 b_1 (e_1^* \wedge e_1^*) + a_1 b_2 (e_1^* \wedge e_2^*) + a_1 b_3 (e_1^* \wedge e_3^*) \\
&\quad + a_2 b_1 (e_2^* \wedge e_1^*) + a_2 b_2 (e_2^* \wedge e_2^*) + a_2 b_3 (e_2^* \wedge e_3^*) \\
&\quad + a_3 b_1 (e_3^* \wedge e_1^*) + a_3 b_2 (e_3^* \wedge e_2^*) + a_3 b_3 (e_3^* \wedge e_3^*) \\
&= (a_1 b_2 - a_2 b_1)(e_1^* \wedge e_2^*) + (a_1 b_3 - a_3 b_1)(e_1^* \wedge e_3^*) + (a_2 b_3 - a_3 b_2)(e_2^* \wedge e_3^*)
\end{aligned}$$

so

$$\Phi(\alpha \wedge \beta) = (a_2 b_3 - a_3 b_2) e_1 - (a_1 b_3 - a_3 b_1) e_2 + (a_1 b_2 - a_2 b_1) e_3$$

which is exactly the cross product of $\Phi(\alpha) \times \Phi(\beta) = (a_1, a_2, a_3) \times (b_1, b_2, b_3)$. □

Problem 3

More generally, the abstract definition of wedge products will imply that if $\varphi_1, \ldots, \varphi_k$ are in $(\mathbb{R}^n)^*$ then

$$\varphi_1 \wedge \ldots \wedge \varphi_k = \sum_{\sigma \in S_k} (-1)^{\operatorname{sgn}(\sigma)} \varphi_{\sigma(1)} \otimes \ldots \otimes \varphi_{\sigma(k)}.$$

In particular, for $\varphi_1, \varphi_2, \varphi_3 \in (\mathbb{R}^3)^*$ we have

$$\begin{aligned} \varphi_1 \wedge \varphi_2 \wedge \varphi_3 = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 - \varphi_1 \otimes \varphi_3 \otimes \varphi_2 \\ + \varphi_2 \otimes \varphi_3 \otimes \varphi_1 - \varphi_2 \otimes \varphi_1 \otimes \varphi_3 \\ + \varphi_3 \otimes \varphi_1 \otimes \varphi_2 - \varphi_3 \otimes \varphi_2 \otimes \varphi_1. \end{aligned}$$

Show that $\{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\}$ is a basis for $\operatorname{Alt}_3(\mathbb{R}^3, \mathbb{R})$.

Proof. We first show that anything of form $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ is alternating:

$$\begin{aligned} (\varphi_1 \wedge \varphi_2 \wedge \varphi_3)(v_1, v_3, v_2) &= \varphi_1(v_1)\varphi_2(v_3)\varphi_3(v_2) - \varphi_1(v_1)\varphi_3(v_3)\varphi_2(v_2) \\ &\quad \varphi_2(v_1)\varphi_3(v_3)\varphi_1(v_2) - \varphi_2(v_1)\varphi_1(v_3)\varphi_3(v_2) \\ &\quad \varphi_3(v_1)\varphi_1(v_3)\varphi_2(v_2) - \varphi_3(v_1)\varphi_2(v_3)\varphi_1(v_2) \\ &= -\varphi_1(v_1)\varphi_2(v_2)\varphi_3(v_3) + \varphi_1(v_1)\varphi_3(v_2)\varphi_2(v_3) \\ &\quad - \varphi_2(v_1)\varphi_3(v_2)\varphi_1(v_3) + \varphi_2(v_1)\varphi_1(v_2)\varphi_3(v_3) \\ &\quad - \varphi_3(v_1)\varphi_1(v_2)\varphi_2(v_3) + \varphi_3(v_1)\varphi_2(v_2)\varphi_1(v_3) \\ &= -(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)(v_1, v_2, v_3). \end{aligned}$$

The remaining cases (e.g. comparing above with input (v_1, v_3, v_2)) are similar and omitted. Since $\mathrm{d}x(p) = e_1^*$ and likewise for the other two, we have

$$(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)(e_1, e_2, e_3) = e_1^*(e_1)e_2^*(e_2)e_3^*(e_3) = 1,$$

so indeed this wedge product is nontrivial.

To show that $\{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\}$ spans $\operatorname{Alt}_3(\mathbb{R}^3, \mathbb{R})$, it suffices to check that the latter is one-dimensional. Indeed, this is true because any multilinear map in $\operatorname{Alt}_3(\mathbb{R}^3, \mathbb{R})$ corresponds to taking the determinant of a 3×3 matrix: if $\varphi_1 = f_{1,1}\mathrm{d}x + f_{1,2}\mathrm{d}y + f_{1,2}\mathrm{d}z$ and likewise for the other two functionals, then

$$\varphi_1 \wedge \varphi_2 \wedge \varphi_3 = \begin{vmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{vmatrix} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

Given that the matrix is fixed and the determinant unique, we see that $\operatorname{Alt}_3(\mathbb{R}^3, \mathbb{R})$ is one-dimensional as any 3-form on $\mathbb{R}^3$ is simply of form $f\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ where f is a function on $\mathbb{R}^3$. $\square$

Problem 4

Prove that $\mathrm{d}\circ\mathrm{d} = 0$ for d acting on 0-forms and 1-forms on $\mathbb{R}^3$.

Proof. For a 0-form f on $\mathbb{R}^3$,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z.$$

It follows that (assuming the order of mixed partials is interchangeable, which is indeed guaranteed by symmetry of $\mathrm{D}^2 f$)

$$\begin{aligned}(\mathrm{d}\circ\mathrm{d})f = &\ \frac{\partial^2 f}{\partial x^2}\mathrm{d}x\wedge\mathrm{d}x + \frac{\partial^2 f}{\partial x\partial y}\mathrm{d}x\wedge\mathrm{d}y + \frac{\partial^2 f}{\partial x\partial z}\mathrm{d}x\wedge\mathrm{d}z \\ &+ \frac{\partial^2 f}{\partial y\partial x}\mathrm{d}y\wedge\mathrm{d}x + \frac{\partial^2 f}{\partial y^2}\mathrm{d}y\wedge\mathrm{d}y + \frac{\partial^2 f}{\partial y\partial z}\mathrm{d}y\wedge\mathrm{d}z \\ &+ \frac{\partial^2 f}{\partial z\partial x}\mathrm{d}z\wedge\mathrm{d}x + \frac{\partial^2 f}{\partial z\partial y}\mathrm{d}z\wedge\mathrm{d}y + \frac{\partial^2 f}{\partial z^2}\mathrm{d}z\wedge\mathrm{d}z \\ = &\left[\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right]\mathrm{d}x\wedge\mathrm{d}y + \left[\frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial z\partial x}\right]\mathrm{d}x\wedge\mathrm{d}z + \left[\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right]\mathrm{d}y\wedge\mathrm{d}z = 0.\end{aligned}$$

Likewise, for any 1-form $\alpha := f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z$ on $\mathbb{R}^3$,

$$\begin{aligned}\mathrm{d}\alpha = &\ \left[\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z\right]\wedge\mathrm{d}x \\ &+ \left[\frac{\partial g}{\partial x}\mathrm{d}x + \frac{\partial g}{\partial y}\mathrm{d}y + \frac{\partial g}{\partial z}\mathrm{d}z\right]\wedge\mathrm{d}y \\ &+ \left[\frac{\partial h}{\partial x}\mathrm{d}x + \frac{\partial h}{\partial y}\mathrm{d}y + \frac{\partial h}{\partial z}\mathrm{d}z\right]\wedge\mathrm{d}z \\ = &\left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right]\mathrm{d}x\wedge\mathrm{d}y + \left[\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right]\mathrm{d}y\wedge\mathrm{d}z + \left[\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right]\mathrm{d}z\wedge\mathrm{d}x,\end{aligned}$$

so taking the exterior derivative once more gives

$$\begin{aligned}(\mathrm{d}\circ\mathrm{d})\alpha = &\ \left[\frac{\partial^2 g}{\partial x\partial z} - \frac{\partial^2 f}{\partial y\partial z}\right]\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \\ &+ \left[\frac{\partial^2 h}{\partial y\partial x} - \frac{\partial^2 g}{\partial z\partial x}\right]\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}x \\ &+ \left[\frac{\partial^2 f}{\partial z\partial y} - \frac{\partial^2 h}{\partial x\partial y}\right]\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}y \\ = &\left[\frac{\partial^2 g}{\partial x\partial z} - \frac{\partial^2 g}{\partial z\partial x} - \frac{\partial^2 f}{\partial y\partial z} + \frac{\partial^2 f}{\partial z\partial y} + \frac{\partial^2 h}{\partial y\partial x} - \frac{\partial^2 h}{\partial x\partial y}\right]\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \\ = &\ 0.\end{aligned}$$

Note that $(x,y,z),(y,z,x)$, and (z,x,y) are all even permutations in S_3 and thus the signs agree. □

Problem 5

Let α be a 1-form on $\mathbb{R}^3$ and let V be the corresponding vector field. Show that the 2-form $\mathrm{d}\alpha$ corresponds to the vector field $\operatorname{curl}(V)$. Similarly, let β be a 2-form on $\mathbb{R}^3$ and let W be the corresponding vector field. Show that the 3-form $\mathrm{d}\beta$ corresponds to the function $\operatorname{div}(W)$.

Proof. The first part directly follows from the previous problem, as the curl of the vector field $\langle f, g, h\rangle$ is precisely

$$\left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle.$$

For the second part, suppose our 2-form looks like $f\mathrm{d}y \wedge \mathrm{d}z + g\mathrm{d}z \wedge \mathrm{d}x + h\mathrm{d}x \wedge \mathrm{d}y$ (so that it represents the vector field $\langle f, g, h\rangle$). Then, its exterior derivative is

$$\begin{aligned}
\mathrm{d}\beta = &\quad \left[\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}y\right] \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&+ \left[\frac{\partial g}{\partial x}\mathrm{d}x + \frac{\partial g}{\partial y}\mathrm{d}y + \frac{\partial g}{\partial z}\mathrm{d}z\right] \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&+ \left[\frac{\partial h}{\partial x}\mathrm{d}x + \frac{\partial h}{\partial y}\mathrm{d}y + \frac{\partial h}{\partial z}\mathrm{d}z\right] \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \frac{\partial f}{\partial x}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \frac{\partial g}{\partial y}\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + \frac{\partial h}{\partial z}\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,
\end{aligned}$$

which indeed corresponds to the divergence of $\langle f, g, h\rangle$. □