HOMEWORK, WEEK 2

This assignment is due Friday, January 29 in lecture. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

(1)

Definition. Let $(A, \preceq)$ be a directed set and let (X, d) be a metric space. A net $f : A \to X$ is *Cauchy* if, for all $\varepsilon > 0$, there exists $a_0 \in A$ such that for all $a_1, a_2 \in A$ with $a_0 \preceq a_1$ and $a_0 \preceq a_2$, we have

$$d(f(a_1), f(a_2)) < \varepsilon.$$

(a) Prove that every convergent net in (X, d) is Cauchy.

(b) Assuming that (X, d) is complete, prove that every Cauchy net in (X, d) converges.

Hint: Given a Cauchy net f, define a sequence in X as follows: choose $a_1 \in A$ such that for $a_1 \preceq a, b$ we have $d(f(a), f(b)) < 1$, and set $x_1 = f(a_1)$. Then choose $a_2 \in A$ with $a_1 \preceq a_2$ such that for $a_2 \preceq a, b$ we have $d(f(a), f(b)) < 1/2$, and set $x_2 = f(a_2)$ (why is this possible?). Continuing in this manner (give a bit more detail), define a sequence $(x_n)_{n=1}^{\infty}$ in X and show it's a Cauchy sequence. Conclude it has a limit $L \in X$, and argue that the net f converges to L.

(c) Let f be a real- or complex-valued function on $[1, \infty)$ such that the improper Riemann integral $\int_1^\infty |f(x)|dx$ exists. Prove that $\int_1^\infty f(x)dx$ exists.

Hint: Let $(A, \preceq) = ([1, \infty), \leq)$ and consider the A-nets in $\mathbb{R}$ defined by $a \mapsto \int_1^a |f(x)|dx$ and $a \mapsto \int_1^a f(x)dx$. Apply the previous parts of the problem along with monotonicity of the integral.

(2) Pugh, Exercise 1.45(a–f) (most of these are short). This problem introduces the limes superior and limes inferior (lim sup and lim inf) which will appear at various points in our discussion of series. For part (d), you can draw pictures if you want but it's not required; give an example of strict inequality in the first ($\leq$) statement and an example where the equality in the second statement does not hold. The formula in part (e) should say that the lim inf of a sequence (a_n) is equal to the lim sup of a different sequence.

(3) This problem explores the properties of the logarithm and exponential functions as rigorously defined using the integral

$$\log(x) = \int_1^x \frac{1}{t}dt.$$

You may assume the following slight generalization of the integration by substitution theorem without proof:

Theorem. *Let u be a diffeomorphism from a compact interval I to a compact interval J with u' Riemann integrable, and suppose I has endpoints $\{a, b\}$. Let f be Riemann integrable on J. Then*

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du,$$

where we follow tradition in using the letter "u" both for a function and for the dummy variable of integration on the right hand side.

This version differs only from the version in the lecture / book in that here we do not assume $a < b$ or that $u' > 0$ (since u is a diffeomorphism we have either $u' > 0$ or $u' < 0$, but $u' < 0$ is allowed here). An integral from a to b for $a > b$ is defined as the negative of the integral from b to a. This version of the integration by substitution theorem can be proved by breaking into cases based on the sign of u' and the sign of $b - a$.

Practically speaking, this generalization means you are free to do u-substitution as you learned in calculus, assuming that u is a diffeomorphism with Riemann integrable derivative. You don't need to check further conditions like $u' > 0$ if you make sure to update the limits of integration appropriately.

(a) Prove that if $x, y \in (0, \infty)$ then $\log(xy) = \log(x) + \log(y)$. *Hint*: By a corollary in Chapter 3.2 of Pugh, you can write the integral $\int_1^{xy} \frac{1}{t}dt$ as

$$\int_1^x \frac{1}{t}dt + \int_x^{xy} \frac{1}{t}dt.$$

The first term is $\log(x)$; use integration by substitution to show that the second term is $\log(y)$.

(b) Prove that if $x \in (0, \infty)$ and n is an integer then $\log(x^n) = n\log(x)$. *Hint*: When n is positive, this follows from the previous part and induction. For n negative, use the first part of the problem to deduce the result from the case of n positive. How about $n = 0$?

(c) Prove that $\log : (0, \infty) \to (-\infty, \infty)$ is surjective. *Hint*: first, show that $\log(2) \geq \frac{1}{2}$. This can be done using monotonicity of the integral (compare $1/t$ to a step function on the interval $[1, 2]$). Then show that $\log(2^n) \geq \frac{n}{2}$, and deduce that the image of log is unbounded above. Show the image of log is also unbounded below, using unboundedness above and the equation $\log(xy) = \log(x) + \log(y)$. Conclude by the intermediate value theorem (why is log continuous?) that the image of log is all of $\mathbb{R}$.

It follows that the inverse function exp to log is defined on all of $\mathbb{R}$.

Definition. Let $e := \exp(1)$ and, for real numbers a, b with $a > 0$, let

$$a^b := \exp(b\log(a)).$$

(d) Prove that $\exp(x) = e^x$ for all $x \in \mathbb{R}$. *Hint:* write out the definition of e^x; what is $\log(e)$, given that $e = \exp(1)$?

(e) Prove that if p and q are integers with $q \neq 0$ and $x \in (0, \infty)$, then

$$\exp((p/q)\log(x))$$

is the unique $y \in (0, \infty)$ with $y^q = x^p$ (both y^q and x^p are repeated multiplication here; existence of y is guaranteed by the intermediate value theorem and uniqueness by the mean value theorem, both applied to $f(x) = x^q$).

Hint: first, use a previous part of the problem to rewrite $p\log(x)$ in a more useful form. Then rewrite x^p as y^q where y is the right hand side of the equation, and use the same previous part to evaluate $\log(y^q)$. You should get that the left hand side equals y as well.

This problem tells us that in the case where exponentiation of real numbers can be understood in terms of repeated multiplication (i.e. the case of rational exponents), our new generalized definition for real exponents agrees with the old one.

(f) For $a \in (1, \infty)$ and $b \in (0, \infty)$, define $\log_a(b) = \frac{\log(b)}{\log(a)}$ (note that $\log = \log_e$). Prove that we have $a^{\log_a(b)} = b$, so that this is a valid definition of base-a logarithms. *Hint*: expand out $a^{\log_a(b)}$ using the definition of exponentiation given above.

(g) Prove that the harmonic series $\sum_{k=1}^{\infty} 1/k$ diverges. *Hint*: use the integral test and the surjectivity of log, proved above.