MATH 425b Problem Set 2

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Cauchy Nets

(1) *This sub-problem defines Cauchy nets and shows some of its properties.*

Definition

*Let $(A, \preceq)$ be a directed set and let (X, d) be a metric space. A net $f : A \to X$ is **Cauchy** if, for all $\epsilon > 0$, there exists $a_0 \in A$ such that for all $a_1, a_2 \in A$ with $a_0 \preceq a_1$ and $a_0 \preceq a_2$, we have*

$$d(f(a_1), f(a_2)) < \epsilon.$$

(a) Prove that every convergent net in (X, d) is Cauchy.

Proof. Let $f : A \to X$ be a convergent net with $\lim f = L$. Let $\epsilon > 0$ be given. We want to show that there exists $a_0 \in A$ such that if $a_0 \preceq a_1, a_2$ then $d(f(a_1), f(a_2)) < \epsilon$.

Indeed, by convergence of f, there exists $a_0' \in A$ such that $d(f(\tilde{a}), L) < \epsilon/2$ for all $\tilde{a} \in A$ satisfying $a_0' \preceq \tilde{a}$. It follows that, for all a_1, a_2 with $a_0' \preceq a_1$ and $a_0' \preceq a_2$, we have

$$d(f(a_1), f(a_2)) \leqslant d(f(a_1), L) + d(L, f(a_2)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and the claim follows. □

(b) Assuming (X, d) is complete, prove that every Cauchy net in (X, d) converges.

Proof. Let $f : A \to X$ be a Cauchy net. We will construct a sequence $\{x_n\}_{n \geqslant 1}$ according to the hint and show f is convergent.

Since f is Cauchy, there exists $a_1 \in a$ such that if $a_1 \preceq a, b$ we have $d(f(a), f(b)) < 1$. Define $x_1 := a_1$. Now pick $a_2 \in A$ such that $a_1 \preceq a_2$ and $(a_2 \preceq a, b \in A \implies d(f(a), f(b)) < 1/2)$. (This is obviously possible if we drop the condition $a_1 \preceq a_2$, and if this binary relation is not automatically fulfilled, i.e., $a_1 \npreceq a_2$, we simply need to use the existence of an upper bound and set it as our new a_2.) Inductively, we may define $a_n \in A$ such that $a_{n-1} \preceq a_n$ and $(a_n \preceq a, b \in A \implies d(f(a) - f(b)) < 1/n)$. Now consider the sequence $\{x_n\}_{n \geqslant 1} := \{f(a_n)\}_{n \geqslant 1}$.

First observation: $\{x_n\}$ converges in X. Indeed, for any given $\epsilon > 0$, for sufficiently large $N \in \mathbb{N}$ with $1/N < \epsilon$, we have $(n, m \geqslant N \implies d(x_n, x_m) < 1/N < \epsilon)$, i.e., it is first Cauchy. Therefore, by the completeness of X we know it converges to some $L \in X$. With this convergence, now we shall show f converges to L as well.

Again let $\epsilon > 0$ be given. Since $\{x_n\}$ converges there exists $N_1 \in \mathbb{N}$ with $d(x_n, L) < \epsilon/2$. On the other hand, there exists $N_2 \in \mathbb{N}$ such that $1/m < \epsilon/2$. Then $(a_{N_2} \preceq a, b \in A \implies d(f(a) - f(b)) < \epsilon/2)$. Now we define $N := \max\{N_1, N_2\}$. It follows that, for *all* $a \in A$ with $a_N \preceq a$,

$$d(f(a) - L) \leqslant d(f(a) - f(a_N)) + d(\underbrace{f(a_N)}_{=x_N} - L) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

from which the claim of convergence of the Cauchy net follows. □

(c) Let f be a real- (or complex-)valued function on $[1, \infty)$ such that the improper Riemann integral $\int_1^\infty |f(x)| \, \mathrm{d}x$ exists. Prove that $\int_1^\infty f(x) \, \mathrm{d}x$ exists.

Proof. According to the hint, we consider the directed set $(A, \preceq) := ([1, \infty), \leqslant)$ and nets $\mathfrak{F}, \overline{\mathfrak{F}}$, both from A to $\mathbb{R}$, defined by

$$\mathfrak{F} : a \mapsto \int_1^a f(x) \, \mathrm{d}x = \text{ and } \overline{\mathfrak{F}} : a \mapsto \int_1^a |f(x)| \, \mathrm{d}x.$$

Let $\epsilon > 0$ be given. By assumption, $\int_1^\infty |f(x)| \, \mathrm{d}x$ exists, so $\overline{\mathfrak{F}}$ converges and is in particular Cauchy. Therefore, there exists $c \in \mathbb{R}$ such that if $a > b > c$,

$$|\overline{\mathfrak{F}}(b) - \overline{\mathfrak{F}}(a)| = \int_a^b |f(x)| \, \mathrm{d}x < \epsilon.$$

Now we want to show that $\mathfrak{F}$ is convergent. By the completeness of $\mathbb{R}$ and results from previous parts it suffices to show that it is Cauchy. Indeed, notice that, for $b > a > c$,

$$|\mathfrak{F}(b) - \mathfrak{F}(a)| = \left| \int_1^b f(x) \, \mathrm{d}x - \int_1^a f(x) \, \mathrm{d}x \right| = \left| \int_a^b f(x) \, \mathrm{d}x \right| \leqslant \int_a^b |f(x)| \, \mathrm{d}x < \epsilon.$$

Therefore $\mathfrak{F}$ is Cauchy and our claim follows. □

Pugh, Ex.1.45, lim sup & lim inf

(2) *Complete Pugh, Exercise 1.45.*

Definition

Let $\{a_n\}_{n \geqslant 1}$ be a sequence of real numbers. It is bounded if the set $A := \{a_1, a_2, \dots\}$ is bounded. The **limit supremum**, or lim sup, of a bounded sequence $\{a_n\}$ as $n \to \infty$ is

$$\limsup_{n \to \infty} a_n = \lim_{n \to \infty} \left(\sup_{k \geqslant n} a_k \right).$$

(a) Why does the lim sup exist?

Solution

The lim sup exists because of the G.L.B. (greatest lower bound) property stated back in Chapter 1. Notice that, if $n_1 \leqslant n_2$, $\sup_{k \geqslant n_1} a_k \geqslant \sup_{k \geqslant n_2} a_k$ (since $n_2 \geqslant n_1$ means we are taking the supremum of fewer terms of the sequence). Hence the sequence $\{s_n\}_{n \geqslant 1}$ defined by $s_n := \sup_{k \geqslant n} a_k$ is monotone decreasing. But since $\{a_n\}$ is bounded by, say $[-M, M]$, $\{s_n\}$ is also bounded below by $-M$. Therefore $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n$ exists.

(b) If $\sup\{a_n\} = \infty$, how should we define $\limsup_{n\to\infty} a_n$? Simply ∞.

(c) If $\lim_{n\to\infty} a_n = -\infty$, how should we define $\limsup_{n\to\infty} a_n$? Simply $-\infty$.

(d) When is it true that

$$\limsup_{n\to\infty}(a_n + b_n) \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

$$\limsup_{n\to\infty} c(a_n) = c \limsup_{n\to\infty} a_n?$$

Solution

Always (assuming we are only looking at bounded sequences). For the first one, since sup of a set is no less than the sup of its subset:

$$\sup_{k \geqslant n}(a_k + b_k) \leqslant \sup_{\substack{j \geqslant n \\ k \geqslant n}}(a_j + b_k) = \sup_{j \geqslant n} a_j + \sup_{k \geqslant n} b_k.$$

As n increases we see $\sup_{k \geqslant n}(a_k + b_k)$ is non-increasing and bounded below, it makes sense to take $\lim_{n\to\infty}$ on both sides, which gives

$$\limsup_{n\to\infty}(a_n + b_n) \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

> **Remark.** For unbounded sequences, if the RHS is not of form $\infty - \infty$, the $\leqslant$ also holds. However, "=" does not always hold: consider $\{a_n\}_{n \geqslant 1} := \{(-1)^n n\}$ and $\{b_n\}_{n \geqslant 1} := \{-a_n\}$. They both have lim sup $+\infty$ but their sum is the constant sequence 0.

For the second one, if $c \geqslant 0$, the claim follows from the fact that if a sequence $\{x_n\} \to x$ then $\{cx_n\} \to cx$. Otherwise, if $c < 0$, since $\limsup_{n\to\infty} -|c|a_n = -\liminf_{n\to\infty}|c|a_n = -|c| \liminf_{n\to\infty} a_n$, we need

$$-|c| \liminf_{n\to\infty} a_n = c \liminf_{n\to\infty} a_n = c \limsup_{n\to\infty} a_n$$

for the original equality to hold. Visualizing such sequences? Think of a sequence with two horizontal lines equidistant to the x-axis (on different sides, of course) as its "asymptotes". (Well, we haven't defined lim inf yet...)

(e) Define the **limit infimum** (lim inf) of a real sequence, and find a formula relating it to lim inf.

Solution

Similar to lim sup, for a bounded sequence $\{a_n\}_{n\geqslant 1}$ we define $\liminf_{n\to\infty} a_n := \lim_{n\to\infty}\left(\inf_{k\geqslant n} a_k\right)$.
In extreme cases:

$$\liminf_{n\to\infty} := \begin{cases} +\infty & \text{if } \lim_{n\to\infty} a_n = \infty, \text{ and} \\ -\infty & \text{if } \inf\{a_n\} = -\infty. \end{cases}$$

A formula that connects lim inf to lim sup:

$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty}(a_n).$$

(f) Prove that $\lim_{n\to\infty} a_n$ exists if and only if the sequence $\{a_n\}$ is bounded and $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Proof. $\implies$: suppose $\lim_{n\to\infty} a_n = L$. We immediately know $\{a_n\}$ is bounded. To show the other equality, we can show that two sides differ by $< \epsilon$ for all $\epsilon > 0$. Since $\{a_n\}$ converges, there exists $N \in \mathbb{N}$ such that $|a_m - L| < \epsilon/4$ for all $m \geqslant N$. Then, taking supremum and infimum on m suggests that the difference is at most $\epsilon/4$ (i.e., $\leqslant \epsilon/4$). Therefore,

$$\sup_{k\geqslant N} a_k - \inf_{k\geqslant N} a_k = \left|\sup_{k\geqslant N} a_k - L\right| + \left|\inf_{k\geqslant N} a_k - L\right| \leqslant \frac{\epsilon}{2} < \epsilon.$$

Recall that $\{s_n\} := \{\sup_{k\geqslant n} a_k\}$ form a monotone decreasing sequence and the sequence of infimums form a monotone increasing sequence. Therefore $\{\Delta_n\} := \{\sup_{k\geqslant n} a_k - \inf_{k\geqslant n} a_k\}$ is also monotone decreasing, so all terms after Δ_n all $< \epsilon$. Since ϵ is arbitrary we conclude that $\Delta_n \to 0$. Hence "$\implies$" follows.

$\impliedby$: suppose $\{a_n\}$ is bounded and its lim inf and lim sup agree, and we define this value to be L. Suppose, for contradiction, that the sequence does *not* converge. Then for some $\epsilon > 0$, there exists a subsequence $\{a_{n_i}\}$ such that $|a_{n_i} - L| \geqslant \epsilon$ for all n_i. WLOG assume infinitely many terms of $\{a_{n_i}\}$ are greater than L, in particular $> L + \epsilon$. Immediately we see that $\limsup_{n\to\infty} a_n \geqslant L + \epsilon$ (since any sequence of form $(x_n, x_{n+1}, \dots)$ contains some element of $\{a_{n_i}\}$ and so the its supremum is at least $L + \epsilon$), a contradiction. Therefore $\{a_n\}$ must converge and converge precisely to L. □

Logarithm and Exponential Functions Done Rigorously

(3) *This problem explores the properties of the logarithm and exponential functions as rigorously defined using*

$$\log(x) = \int_1^x \frac{1}{\tilde{t}}\,\mathrm{d}\tilde{t}.$$

Theorem: a slight generalization of integration by substitution

Let u be a diffeomorphism from a compact interval I to a compact interval J with u' Riemann integrable, and suppose I has endpoints $\{a,b\}$. Let f be Riemann integrable on J. Then

$$\int_a^b f(u(\tilde{x}))u'(\tilde{x})\,\mathrm{d}\tilde{x} = \int_{u(a)}^{u(b)} f(\tilde{u})\,\mathrm{d}\tilde{u}.$$

(a) Prove that if $x, y \in (0,\infty)$ then $\log(xy) = \log(x) + \log(y)$.

Proof. (By the hint:)

$$\begin{aligned}\log(xy) = \int_1^{xy} \frac{1}{\tilde{t}}\,\mathrm{d}\tilde{t} &= \int_1^{x} \frac{1}{\tilde{t}}\,\mathrm{d}\tilde{t} + \int_x^{xy} \frac{1}{\tilde{t}}\,\mathrm{d}\tilde{t} && \text{(Pugh, Cor.3.2.30)}\\ &= \log(x) + \int_1^{y} \frac{1}{\tilde{t}} \cdot 1\,\mathrm{d}\tilde{t} && \text{(substituting } u(t) := t\text{)}\\ &= \log(x) + \log(y).\end{aligned}$$

□

(b) Prove that if $x \in (0,\infty)$ and n is an integer then $\log(x^n) = n\log(x)$.

Proof. (By the hint:) if $n > 0$ then setting $y := x$ from (a) and using it inductively proves the claim. If $n = 0$, by definition

$$\log(x^0) = \log(1) = \int_1^1 \frac{1}{\tilde{t}}\,\mathrm{d}\tilde{t} = 0 = 0x.$$

If $n < 0$, consider $\log(x^n x^{-n}) = \log(1) = 0$, while (a) suggests that

$$0 = \log(x^n x^{-n}) = \log(x^n) + \underbrace{\log(x^{-n})}_{-n\log(x)} \implies \log(x^n) = n\log(x) \text{ for } n < 0 \text{ as well.}$$

□

(c) Prove that $\log : (0,\infty) \to (-\infty,\infty)$ is surjective.

Proof. (By the hint:) first notice that $\log(2) = \int_1^2 \frac{1}{\tilde{t}}\,\mathrm{d}\tilde{t} \geqslant \int_1^2 \frac{1}{2}\,\mathrm{d}\tilde{t} = (2-1)\cdot\frac{1}{2} = \frac{1}{2}$. Therefore by (b), if $n > 0$, we have

$$\log(2^n) \overset{\text{(b)}}{=} n\int_1^2 \frac{1}{\tilde{t}}\,\mathrm{d}\tilde{t} \geqslant \frac{n}{2}.$$

Letting $n \to \infty$ we see that $\log(2^n) \to \infty$, and so log is unbounded from above. From (b) we also know that $\log(2^n 2^{-n}) = 0 = \log(2^n) + \log(2^{-n})$ so, for $n > 0$, $\log(2^{-n}) = -\log(2^n)$. Therefore as $n \to \infty$ we have $\log(2^{-n}) \to -\infty$ and so log is unbounded from below as well.

Now, it remains to notice that, by FTC (part I), since $1/\tilde{t}$ is continuous everywhere for $t \neq 0$, in particular $t > 0$, the log function is continuous on $(0,\infty)$. Since anything of form $\log(2^n)$ or $\log(2^{-n})$ are on this domain, by IMT it follows that the image of log is unbounded both from below and above, and it surjects onto $\mathbb{R}$. □

Definition

Let $\exp =: (-\infty, \infty) \to (0, \infty)$ *be the inverse function to* $\log$*. Let* $e := \exp(1)$ *and, for real numbers* a, b *with* $a > 0$*, define*

$$a^b := \exp(b \log(a)).$$

(d) Prove that $\exp(x) = e^x$ for all $x \in \mathbb{R}$.

Proof. (By the hint:) By definition, since $e = \exp(1) > 0$, $e^x = \exp(x \log(e)) = \exp(x \cdot \log \exp(1)) = \exp(x)$. □

(e) Prove that if p and q are integers with $q \neq 0$ and $x \in (0, \infty)$, then

$$\exp((p/q)\log(x))$$

is the unique $y \in (0, \infty)$ with $y^q = x^p$.

Proof. (By the hint:) since $f(\tilde{x}) = \tilde{x}^q$ is continuous and monotone on $(0, \infty)$, IMT and MVT suggests the existence and uniqueness of solution of y to $y^q = x^p$. Therefore it suffices to show the above expression is a solution. Indeed,

$$\begin{aligned}
[\exp((p/q)\log(x))]^q &= \exp\left[q \cdot \log \exp[(p/q)\log(x)]\right] && \text{(definition of exp)} \\
&= \exp\left[q \cdot (p/q)\log(x)\right] \\
&= \exp\left[p \log(x)\right] = \exp \log(x^p) = x^p.
\end{aligned}$$

□

(f) For $a \in (1, \infty)$ and $b \in (0, \infty)$, define

$$\log_a(b) := \frac{\log(b)}{\log(a)}.$$

Prove that we have $a^{\log_a(b)} = b$ so that this is a valid definition of base-a logarithms.

Proof. (By the hint:) expanding $a^{\log_a(b)}$ using the definition, we have

$$a^{\log_a(b)} = \exp\left[\log_a(b)\log(a)\right] = \exp\left[\frac{\log(b)}{\log(a)}\log(a)\right] = \exp \log(b) = b.$$

□

(g) Prove that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Proof. (By the hint:) we consider the integral test between $\sum_{k=1}^{\infty} \frac{1}{k}$ and $\int_1^{\infty} \frac{1}{\tilde{t}} \, \mathrm{d}\tilde{t}$. Notice that we can treat $\sum_{k=1}^{\infty} \frac{1}{k}$ as $\int_1^{\infty} \frac{1}{\lfloor \tilde{t} \rfloor} \, \mathrm{d}\tilde{t}$ which dominates $\int_1^{\infty} \frac{1}{\tilde{t}} \, \mathrm{d}\tilde{t}$ since the floor function $\lfloor \tilde{t} \rfloor \leqslant t$. However, we also know

$$\int_1^{\infty} \frac{1}{\tilde{t}} \, \mathrm{d}\tilde{t} = \lim_{n \to \infty} \int_1^n \frac{1}{\tilde{t}} \, \mathrm{d}\tilde{t} = \lim_{n \to \infty} \log(n) \to \infty$$

as log surjects onto $\mathbb{R}$. Therefore the harmonic series diverges. □