HOMEWORK, WEEK 3

This assignment is due Friday, February 5 in lecture. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

(1) Pugh, Exercise 3.69. This exercise shows that you can rearrange the terms of an absolutely convergent series without affecting convergence or the sum. For part (a), note that given $\varepsilon > 0$, you can ensure that the terms $a_{N+1}, a_{N+2}, \ldots$ contribute less than ε to the sum. Show that you can choose N' large enough that all N of the terms $a_1, \ldots, a_N$ appear as terms $a_{\beta(1)}, \ldots, a_{\beta(N')}$. Using similar arguments, show that no partial sum of the rearranged series can be larger than the sum of the original infinite series. For part (b), note that the series of absolute values of a rearrangement is a rearrangement of the original series of absolute values.

(2) Let $\sum_{k=0}^{\infty} c_k(x-x_0)^k$ be a real or complex power series with radius of convergence R. In this problem you will show that the radius of convergence of the term-by-term differentiated series $\sum_{k=1}^{\infty} kc_k(x-x_0)^{k-1}$ and the term-by-term integrated series $\sum_{k=0}^{\infty} \frac{c_k}{k+1}(x-x_0)^{k+1}$ are also R; this is part of the proof of Theorem 4.12 in the book, but here you'll fill in the details. This is an important proof in both real and complex analysis.

(a) Suppose a_n, b_n are sequences of real numbers with $a_n \geq 0$ and $b_n \geq 0$ for all n. Show that $\sup a_n b_n \leq \sup a_n \sup b_n$, as long as the right side of the inequality is not an indeterminate form $0 \times \infty$ or $\infty \times 0$.

Hint: For a fixed n, show that $a_n b_n$ is less than or equal to the product of suprema (which is well-defined by assumption).

(b) Suppose a_n, b_n are sequences of real numbers with $a_n \geq 0$ and $b_n \geq 0$ for all n. Show that

$$\limsup a_n b_n \leq \limsup a_n \limsup b_n,$$

as long as the right side of the inequality is not an indeterminate form $0 \times \infty$ or $\infty \times 0$.

Hint: Use the definition of lim sup as the limit of a supremum on the left side. Use the previous problem (show that the assumptions are satisfied), then use that the limit of a product sequence is the product of limits (you can assume that this holds under the given assumptions; the finite cases follow from continuity of the multiplication map from $\mathbb{R}^2$ to $\mathbb{R}$).

(c) Suppose a_n, b_n are sequences of real numbers with $a_n \geq 0$ and $b_n \geq 0$ for all n. Assume that $\lim_{n\to\infty} a_n = A$ with $0 < A < \infty$, and write $B = \limsup b_n \in [0, \infty]$. Show that

$$\limsup a_n b_n = AB.$$

Hint: Since you have $\limsup a_n = \lim a_n$ whenever the limit exists, the previous problem gives you $\limsup a_n b_n \leq AB$. To show the reverse inequality, consider two cases. When $B < \infty$, show that for all $\varepsilon > 0$, you have $\limsup a_n b_n \geq (A-\varepsilon)B$. To do this, given ε, use the convergence of a_n to A to bound a_n below by $A - \varepsilon$ for large enough n.

When $B = \infty$, the proof is similar but a bit simpler. It suffices to show that $\limsup a_n b_n \geq M$ for all M. Given a fixed M, use convergence of a_n to A to bound a_n below by some fixed positive number (say $A/2$) for all large n, then choose a large n' such that $b_{n'} \geq 2M/A$.

(d) Show that $\lim_{k\to\infty} k^{1/k} = 1$ and $\lim_{k\to\infty}(\frac{1}{k+1})^{1/k} = 1$.

Hint: Write $k^{1/k}$ as $e^{(1/k)\log(k)}$, and try to evaluate the limit of the exponent. For the second limit, you can argue similarly; what does $\log(\frac{1}{k+1})$ equal?

(e) Show that

$$\limsup \sqrt[k]{k|c_k|} = \limsup \sqrt[k]{|c_k|} = \limsup \sqrt[k]{\frac{|c_k|}{k+1}}.$$

Hint: Write $(k|c_k|)^{1/k}$ as $k^{1/k}|c_k|^{1/k}$ and use the previous problems; the second equality is similar.

(f) Show that the differentiated series

$$\sum_{k=1}^{\infty} k c_k (x - x_0)^{k-1}$$

and the integrated series

$$\sum_{k=0}^{\infty} \frac{c_k}{k+1}(x - x_0)^{k+1}$$

have the same radius of convergence R as the original series $\sum_{k=0}^{\infty} c_k (x - x_0)^k$.

Hint: The book's proof in Theorem 4.12 features a complicated lim sup of an exponential, which you can avoid as follows: first show that $\sum_{k=1}^{\infty} k c_k (x-x_0)^{k-1}$ and $\sum_{k=1}^{\infty} k c_k (x - x_0)^k$ have the same radius of convergence. Show the same thing for $\sum_{k=0}^{\infty} \frac{c_k}{k+1}(x - x_0)^{k+1}$ and $\sum_{k=0}^{\infty} \frac{c_k}{k+1}(x - x_0)^k$. Then use the previous problem to finish your proof.