MATH 425b Homework 3

Qilin Ye

January 30, 2021

Pugh, Ex.3.69

Consider a series $\sum_{k=1}^{\infty} a_k$ and **rearrange** its terms by some bijection $\beta : \mathbb{N} \to \mathbb{N}$, forming a new series $\sum_{k=1}^{\infty} a_{\beta(k)}$.

(a) Prove that every rearrangement of a convergent series of nonnegative terms converges and converges to the same sum as the original series.

> *Proof.* Let $\epsilon > 0$ be given. By CCC there exists $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} a_k < \epsilon$. Notice that $\{a_1, \dots, a_N\}$ gets mapped to $\{a_{\beta(1)}, \dots, a_{\beta(N)}\}$ under the rearrangement. Define $N' := \max_{n \leqslant N} \beta(n)$. This is well-defined since N is finite. It follows that $N' \geqslant N$, so $\{a_{N'+1}, a_{N'+2}, \dots\} \subset \{a_{N+1}, a_{N+2}, \dots\}$. Therefore, $\sum_{k=N'+1}^{\infty} a_{\beta(k)} \leqslant \sum_{k=N+1}^{\infty} a_k < \epsilon$, and this shows precisely the convergence of $\sum_{k=1}^{\infty} a_{\beta(k)}$ according to CCC.
>
> From above we see that the rearranged series can have partial sums arbitrarily close to the original sum from below. Therefore it suffices to show that no rearranged partial sum exceeds the original limit. Indeed, for all $n' \in \mathbb{N}$, there exists a sufficiently large $n \in \mathbb{N}$ such that $\{\beta(1), \dots, \beta(n')\} \subset \{1, 2, \dots, n\}$. Since a_k's are all nonnegative, $\sum_{k=1}^{n'} a_{\beta(k)} \leqslant \sum_{k=1}^{n} a_k \leqslant \sum_{k=1}^{\infty} a_k$, and our claim follows. □

(b) Do the same for absolutely convergent series.

> *Proof.* This is already given by the hint... The series of absolute values of a rearrangement is the rearrangement of the original series of absolute values, so if the original series is absolutely convergent, so is the rearranged series. □

Term-by-Term Differentiation & Integration of Power Series

Let $\sum_{k=0}^{\infty} c_k (x - x_0)^k$ be a real or complex power series with radius of convergence R. This problem shows that the radii of convergence of the term-by-term differentiated series and of the term-by-term integrated series are also R.

(a) Suppose $\{a_n\}, \{b_n\}$ are sequences of real, nonnegative numbers. Show that $\sup a_n b_n \leqslant \sup a_n \sup b_n$ as long as the RHS is not an indeterminate form $0 \times \infty$ or $\infty \times 0$. Immediate since $\sup_{n \geqslant 1} a_n b_n = \sup_{n \geqslant 1} a_n b_n \leqslant \sup_{n \geqslant 1} a_n \sup_{m \geqslant 1} b_m$.

(b) Suppose a_n, b_n are sequences of real, nonnegative numbers. Show that

$$\limsup a_n b_n \leqslant \limsup a_n \limsup b_n$$

as long as the RHS is not an indeterminate form $0 \times \infty$ or $\infty \times 0$.

Proof. We simply need to apply $\lim_{n\to\infty}$ to (a):

$$\begin{aligned}\limsup a_n b_n &= \lim_{n\to\infty}\left(\sup_{k\geqslant n} a_k b_k\right) \leqslant \lim_{n\to\infty}\left(\sup_{k\geqslant n} a_k \sup_{j\geqslant n} b_j\right)\\ &= \lim_{n\to\infty}\left(\sup_{k\geqslant n} a_k\right)\lim_{n\to\infty}\left(\sup_{j\geqslant n} b_j\right)\\ &= \limsup a_n \limsup b_n.\end{aligned}$$

□

(c) Further assume that $\lim_{n\to\infty} a_n = A \in (0,\infty)$ and $\limsup_{n\to\infty} b_n = B \in [0,\infty]$. Show that

$$\limsup a_n b_n = AB.$$

Proof. Since the convergence of a_n implies $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = A$, by (b) we already have $\leqslant$. It remains to show $\geqslant$.

By the convergence of a_n, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geqslant N \implies a_n \geqslant A - \epsilon.$$

Hence if $m \geqslant N$ we have $\sup_{k\geqslant m} a_k b_k \geqslant \sup_{k\geqslant m}(A-\epsilon)b_k = (A-\epsilon)\sup_{k\geqslant m} b_k$. Taking $m \to \infty$ gives

$$\limsup_{m\to\infty} a_m b_m \geqslant (A-\epsilon)\limsup_{m\to\infty} b_m = (A-\epsilon)B.$$

Since ϵ is arbitrary, we indeed have $\limsup a_n b_n \geqslant AB$ and thus $\limsup a_n b_n = AB$. □

(d) Show that $\lim_{k\to\infty} k^{1/k} = 1$ and $\lim_{k\to\infty}(1/(k+1))^{1/k} = 1$.

Proof. Writing $k^{1/k}$ as $\exp((1/k)\log(k))$ we have

$$\lim_{k\to\infty}\frac{\log(k)}{k} \overset{(H)}{=} \lim_{k\to\infty}\frac{1/k}{1} = 0$$

and so $\lim_{k\to\infty}\exp((1/k)\log(k)) = e^0 = 1$ (since exp is continuous).

Similarly, since

$$\left(\frac{1}{k+1}\right)^{1/k} = \exp\left(\frac{1}{k}\log\left(\frac{1}{k+1}\right)\right) = \exp\left(\frac{1}{k}\Big[\underbrace{\log(1)}_{=0} - \log(k+1)\Big]\right),$$

taking $k \to \infty$ we see that

$$\lim_{k\to\infty} -\frac{\log(k+1)}{k} \overset{(H)}{=} \lim_{k\to\infty} -\frac{1}{k+1} = 0,$$

and again $\lim_{k\to\infty}(1/(k+1))^{1/k} = e^0 = 1$. □

(e) Show that

$$\limsup_{k\to\infty}\sqrt[k]{k|c_k|} = \limsup_{k\to\infty}\sqrt[k]{|c_k|} = \limsup_{k\to\infty}\sqrt[k]{\frac{|c_k|}{k+1}}.$$

Proof. Using (c) and (d), we have

$$\begin{aligned}\limsup_{k\to\infty} \sqrt[k]{|c_k|} &= \limsup_{k\to\infty} k^{1/k}|c_k|^{1/k} \\ &= \limsup_{k\to\infty} k^{1/k} \limsup_{k\to\infty} |c_k|^{1/k} && \text{(by (c))} \\ &= \limsup_{k\to\infty} |c_k|^{1/k}, && \text{(by (d))}\end{aligned}$$

and

$$\begin{aligned}\limsup_{k\to\infty} \sqrt[k]{\frac{|c_k|}{k+1}} &= \limsup_{k\to\infty} \left(\frac{1}{k+1}\right)^{1/k} |c_k|^{1/k} \\ &= \limsup_{k\to\infty} \left(\frac{1}{k+1}\right)^{1/k} \limsup_{k\to\infty} |c_k|^{1/k} && \text{(by (c))} \\ &= \limsup_{k\to\infty} |c_k|^{1/k}. && \text{(by (d))}\end{aligned}$$

□

(f) Show that the differentiated series $\sum_{k=1}^{\infty} kc_k(x-x_0)^{k-1}$ and the integrated series $\sum_{k=0}^{\infty} \frac{c_k}{k+1}(x-x_0)^{k+1}$ both have radius of convergence R as the original series $\sum_{k=0}^{\infty} c_k(x-x_0)^k$.

Proof. For the first one, note that $\sum_{k=1}^{\infty} kc_k(x-x_0)^{k-1}$ can be rewritten as $\sum_{k=0}^{\infty}(k+1)c_{k+1}(x-x_0)^k$. Then,

$$\begin{aligned}\limsup_{k\to\infty} \sqrt[k+1]{|(k+1)c_{k+1}|} &= \limsup_{k\to\infty} \sqrt[k+1]{(k+1)} \limsup_{k\to\infty} \sqrt[k+1]{|c_{k+1}|} \\ &= \limsup_{k\to\infty} k^{1/k} \limsup_{k\to\infty} |c_k|^{1/k} \\ &= \limsup_{k\to\infty} |c_k|^{1/k}.\end{aligned}$$

Therefore the differentiated series has radius of convergence R. Similarly, notice that

$$\limsup_{k\to\infty} \sqrt[k]{\frac{c_k}{k+1}} = \limsup_{k+1\to\infty} \sqrt[k+1]{\frac{c_{k+1}}{k+2}}.$$

Hence $\sum_{k=0}^{\infty} \frac{c_k}{x+1}(x-x_0)^{k+1}$ and $\sum_{k=0}^{\infty} \frac{c_{k+1}}{k+2}(x-x_0)^{k+1} = \sum_{k=1}^{\infty} \frac{c_k}{k+1}(x-x_0)^k$ have the same radius of convergence. The main claim follows since the latter has the same radius of convergence as the original one by (e). □