HOMEWORK, WEEK 4

This assignment is due Friday, February 12 in lecture. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

(1) Pugh, Exercise 4.4(b). This exercise investigates whether uniform continuity is preserved by uniform convergence; you only need to do part (b), because part (a) follows as a special case.

Hint: The answer is "yes," uniform convergence preserves uniform continuity, even in the setting of arbitrary metric spaces. The proof is a simple variation of the $\varepsilon/3$ proof from Friday's lecture that a uniform limit of continuous functions is continuous (Theorem 1 in Section 4.1 of Pugh).

(2) Prove the following result about uniform convergence.

Theorem (Dini's theorem)**.** *Let X be a compact metric space and let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions. Assume that the functions f_n converge pointwise to $f : X \to \mathbb{R}$, that f is continuous, and that for all $x \in \mathbb{R}$, the sequence $(f_n(x))_{n=1}^{\infty}$ is a decreasing sequence of real numbers (the same result would hold if these sequences were always increasing). Then f_n converges uniformly to f.*

Hint: Note that you have $f_n(x) \geq f(x)$ for all x. Given $\varepsilon > 0$, you want to find N such for $n \geq N$, you have $f_n(x) - f(x) < \varepsilon$ for all $x \in X$. For each n, you have an open (why?) subset U_n of X consisting of points where this inequality is satisfied; you want to show that some U_n is equal to all of X.

To do this, first show that the U_n cover X; you should be able to do this using pointwise convergence of the f_n to f. Since X is compact, you can extract a finite subcover. Finally, show that for $n < n'$ we have $U_n \subset U_{n'}$ using the decreasing property assumed in the statement. From there, you should be able to find U_n with $U_n = X$, i.e. $f_n(x) - f(x) < \varepsilon$ for all $x \in X$, proving uniform convergence.

(3) Let A be a directed set. Consider a sequence $f_n(a)$ of nets from A to a metric space Y. Assume that:

- For each fixed n, the net $f_n(a)$ converges to a limit L_n in Y.
- The nets f_n converge uniformly (as a sequence of functions $A \to Y$) to some net $f : A \to Y$.

(a) Prove that the limits $(L_n)_{n=1}^{\infty}$ form a Cauchy sequence in Y.

Hint: This is a more elaborate version of the "$\varepsilon/3$" argument. Given $\varepsilon > 0$, you want N such that $d(L_n, L_m) < \varepsilon$ for all $n, m \geq N$. Choose N such that for $n, m \geq N$, we have $d(f_n(a), f_m(a)) < \varepsilon/3$ for all $a \in A$ (why is this possible?).

You want to show that if $n, m \geq N$ where N was chosen above, then $d(L_n, L_m) < \varepsilon$. To do this, consider fixed integers $n, m \geq N$. Use the assumptions $\lim f_n = L_n$ and $\lim f_m = L_m$ to find a_0 such that $d(f_n(a), L_n) < \varepsilon/3$ and $d(f_m(a), L_m) < \varepsilon/3$

for all $a_0 \preceq a$ (which directed set axiom are you using?). Then use the triangle inequality as in the usual $\varepsilon/3$ argument.

(b) Now assume that the sequence $(L_n)_{n=1}^{\infty}$ converges to some limit $L \in Y$ (for example, this will always hold if Y is complete, by the previous part). Prove that the limit of f (as a net from A to Y) exists and equals L.

Hint: To build an argument like this, one way is to work backwards from the triangle inequality. Given $\varepsilon > 0$, the inequality you want is $d(f(a), L) < \varepsilon$ (for all $a_0 \preceq a$, some a_0), so you want a chain of controllable quantities connecting $f(a)$ and L. What you know about L is that it's the limit of the sequence (L_n), so you're expecting to at least use $d(f(a), L) \leq d(f(a), L_n) + d(L_n, L)$. The quantity $d(L_n, L)$ is independent of a; you can make it small by choosing a (fixed) n that's large enough.

For this fixed n, you also want to control $d(f(a), L_n)$, and it's reasonable to try $d(f(a), L_n) \leq d(f(a), f_n(a)) + d(f_n(a), L_n)$. Since n is fixed, the second quantity is controlled for $a_0 \preceq a$ (for some a_0). For the first quantity, you might worry that the convergence of f_n to f depended on the point a, so that your choice of n here would depend on your choice of a (which in turn depended on your fixed n above). This would indeed be a problem if f_n only converged pointwise to f, but luckily you have uniform convergence, so everything is fine.

Now go back and rearrange this reasoning in the logically correct order, which may involve moving some things around. Given ε, the logical dependence is that N should depend on ε, and then a_0 should depend on N. So you want to choose N, satisfying all the conditions it'll eventually need to satisfy, first. Once this N is chosen, then you choose a_0, satisfying all the things it needs to satisfy. Then given $a_0 \preceq a$, use the triangle inequality and assumptions to prove what you want. (Note that you don't need to consider $n \geq N$; you can just work with f_N and L_N.)

(c) Use the above part to give an alternate proof that if X and Y are metric spaces and $f_n : X \to Y$ are continuous at $x \in X$ and converge uniformly to $f : X \to Y$, then f is continuous at x.

Hint: The idea is that "uniform convergence preserves limits" should imply "uniform convergence preserves continuity." For $x \in X$, there are two possibilities. First, if x is an isolated point (i.e. $x \notin \overline{X \setminus \{x\}}$, i.e. $\{x\}$ is an open subset of X), show that any function $f : X \to Y$ is continuous at x.

The second possibility is that x is a non-isolated point. In this case, consider the directed set $A = X \setminus \{x\}$ with ordering relation $a \preceq a'$ if $|a' - x| \leq |a - x|$. Each function f_n can be restricted to a function from $A = X \setminus \{x\}$ to Y, i.e. a net from A to Y.

Assuming x is a non-isolated point (i.e. $x \in \overline{X \setminus \{x\}}$), first show that a function $g : X \to Y$ is continuous at $x \in X$ if and only if the restriction of g to $A = X \setminus \{x\}$, viewed as a net from A to Y, converges (as a net) to $g(x)$. Then you know that the limit of f_n (as a net $A \to Y$) exists and equals $f_n(x)$. Why does the sequence $(f_n(x))_{n=1}^{\infty}$ converge, and what is its limit? Conclude by the above part that the limit of f (as a net $A \to Y$) exists and equals $f(x)$, so f is continuous at x.

Remark. It's true that the direct $\varepsilon/3$ argument we saw in class is much shorter than the above alternate proof that uniform convergence preserves continuity. It's still of conceptual interest, though. In calculus, limits are often presented as the fundamental concept from which continuity, derivatives, etc. can be defined (this approach runs into difficulties with Riemann integrals unless you generalize the meaning of "limit," for example by using nets). Based on this perspective, it's natural to ask whether you can first prove that "uniform convergence preserves limits of nets" (an informal statement of Problem 2b), and then deduce results on uniform convergence and many different calculus constructions. The answer is yes in general; the next problem gives another example.

(d) Use part (3b) to give an alternate proof that if $[a, b]$ is a closed interval in $\mathbb{R}$ and $f_n : [a, b] \to \mathbb{R}$ are Riemann integrable functions converging uniformly to $f : [a, b] \to \mathbb{R}$, then f is Riemann integrable.

Hint: Let A be the set of tagged partitions (P, T) of $[a, b]$ with the refinement ordering $\preceq$ that was discussed in HW 1. By that homework, you know that Riemann integrability of a function $g : [a, b] \to \mathbb{R}$ is equivalent to convergence of the net $(P, T) \mapsto R(g, P, T)$ (a net from A to $\mathbb{R}$).

You want to show that the nets $(P, T) \mapsto R(f_n, P, T)$ converge uniformly in (P, T) (as $n \to \infty$) to the net $(P, T) \mapsto R(f, P, T)$. Since $\mathbb{R}$ is complete, the result will then follow from the first two parts of the problem.

To prove this uniform convergence of nets, let $\varepsilon > 0$ be given. You want N such that for $n \geq N$, we have $|R(f, P, T) - R(f_n, P, T)| < \varepsilon$ for all $n \geq N$ and all partition pairs (P, T). Use uniform convergence of f_n to f to choose N such that for $n \geq N$, we have $|f(x) - f_n(x)| < \frac{\varepsilon}{b-a}$ for all $x \in [a, b]$; from here you should be able to derive the desired inequality.