MATH 425b Problem Set 4

Qilin Ye

February 11, 2021

Pugh, Ex. 4.4(b)

Problem

Show that if $f_n : (X, d) \to (Y, d')$ is uniformly continuous for each $n \in \mathbb{N}$ and if $f_n \to f$ uniformly as $n \to \infty$, then f is uniformly continuous.

Proof. Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that if $d(x, y) < \delta$ then $d'(f(x), f(y)) < \epsilon$.
Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that $d'(f_n(\tilde{x}) - f(\tilde{x})) < \epsilon/3$ for all $\tilde{x} \in X$ whenever $n \geqslant N$. Fix any n greater than N. On the other hand, since f_n is uniformly continuous, there also exists $\delta > 0$ such that $d'(f_n(\tilde{x}), f_n(\tilde{y})) < \epsilon/3$ whenever $d(\tilde{x}, \tilde{y}) < \delta$. Combining the two results with triangle inequality, we have

$$\begin{aligned} d'(f(x), f(y)) &\leqslant d'(f(x), f_n(x)) + d'(f_n(x), f_n(y)) + d'(f_n(y), f(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ whenever } d(x, y) < \delta \text{ for } x, y \in X. \end{aligned}$$

Hence f is uniformly continuous. □

Dini's Theorem

Prove the following:

Theorem: Dini's Theorem

Let X be a compact metric space and let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions. Assume that the functions f_n converge pointwise to $f : X \to \mathbb{R}$, that f is continuous, and that for all $x \in X$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is a decreasing sequence of real numbers. Then f_n converges uniformly to f. Same result holds if $\{f_n(x)\}_{n=1}^{\infty}$ is a monotone increasing function.

Proof. Let $\epsilon > 0$ be given. Our goal is find $N \in \mathbb{N}$ such that $|f_n(x), f(x)| < \epsilon$ for all $x \in X$ whenever $n \geqslant N$.
Define $g_n := f_n - f$; it follows that g_n is continuous function from X to $[0, \infty)$. Define $U_n := g_n^{-1}\{[0, \epsilon)\}$. By open set condition we know each U_n is open. Since $f_n \to f$ pointwise, each $x \in X$ is contained in some U_i for i sufficiently large. Therefore

$$\bigcup_{i=1}^{\infty} U_i \text{ is an open cover of } X.$$

By the compactness of X we can extract a finite subcover $\bigcup_{i=1}^{k} U_i \supset X$. Since $\{f_n\}$ is monotonously decreasing, so is $\{g_n\}$, and thus $\{U_n\}$ is increasing. Therefore if the subcover covers X, so does U_k the largest scrap itself. Hence if $n \geqslant k$, $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$, i.e., $f_n \to f$ uniformly. □

More on Nets

Let A be a directed set. Consider a sequence $f_n(a)$ of nets from A to a metric space Y. Assume that:

(1) For each fixed n, the net $f_n(a)$ converges to a limit L_n in Y.

(2) The nets f_n converge uniformly (as a sequence of functions $A \to Y$) to some net $f : A \to Y$.

(a) Prove that the limits $\{L_n\}_{n \geqslant 1}^{\infty}$ form a Cauchy sequence in Y.

Proof. Let $\epsilon > 0$ be given. Our goal is to show that $d(L_m, L_n) < \epsilon$ for sufficiently large m, n.
Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that $d(f_n(\tilde{a}), f_m(\tilde{a})) < \epsilon/3$ for all $\tilde{a}$ whenever $m, n \geqslant N$.
Now pick and fix m, n. Since $\lim f_n = L_n$, there exists $a_n \in A$ such that $d(f_n(\tilde{a}), L_n) < \epsilon/3$ whenever $a_n \leq \tilde{a}$.
Likewise, since $\lim f_m = L_m$, there exists $a_m \in A$ such that $d(f_m(\tilde{a}), L_m) < \epsilon/3$ whenever $a_m \leq \tilde{a}$.
By the third axiom of directed set, let $\tilde{a} \in A$ be an upper bound for a_n and a_m. Then for all $a \in A$ with $\tilde{a} \leq a$, we have

$$\begin{aligned} d(L_m, L_n) &\leqslant d(L_n, f_n(a)) + d(f_n(a), f_m(a)) + d(f_m(a), L_m) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

i.e., the limits form a Cauchy sequence in Y. □

(b) Now assume that the sequence $\{L_n\}_{n \geqslant 1}^{\infty}$ converges to some $L \in Y$. *For example, this will always hold if Y is complete.* Prove that the limit of f (as a net from A to Y) exists and equals L.

Proof. Let $\epsilon > 0$ be given. Our goal is to show that $d(f(a), L) < \epsilon$ for all a with $a_0 \leq a$ for some a_0.
Since $f_n \to f$ uniformly, there exists $N_1 \in \mathbb{N}$ such that $d(f(\tilde{a}), f_n(\tilde{a})) < \epsilon/3$ for all $\tilde{a}$ whenever $n \geqslant N$.
Since $L_n \to L$, there exists another $N_2 \in \mathbb{N}$ such that $d(L_n, L) < \epsilon/3$ whenever $n \geqslant N$.
Finally, since $\lim f_n = L_n$, there exists $a_n \in A$ such that $d(f_n(\tilde{a}), L_n) < \epsilon/3$ whenever $a_n \leq \tilde{a}$.
Now combine what we have above. Define $N := \max\{N_1, N_2\}$. Let $a \in A$ be such that $a_n \leq a$. Then,

$$\begin{aligned} d(f(a), L) &\leqslant d(f(a), f_N(a)) + d(f_N(a), L_N) + d(L_N, L) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

since a is arbitrary as long as $a_n \leq a$, we conclude that $f(a) \to L$, i.e., $\lim f = L$. □

(c) Use the above part to give an alternate proof that if X and Y are metric spaces and $f_n : X \to Y$ are continuous at $x \in X$ and converge uniformly to $f : X \to Y$, then f is continuous at x.

Proof. Pick $x \in X$. If x is an isolation point, then if a sequence $\{x_n\} \to x$, the tail is constant. Therefore any function $X \to Y$, including f, maps $\{x_n\}$ to a sequence with a constant tail in Y, continuous indeed.
If $x \in X$ is not an isolation point, define a directed set $A := X \setminus \{x\}$ with $\leq$: $a \leq a'$ if $d(a' - x) \leqslant d(a - x)$.
We first prove a lemma below:

Lemma

If $x \in X$ is not isolated, then $g : X \to Y$ is continuous at x if and only if $g\big|_A$ viewed as a net to $\mathbb{R}$ converges to $g(x)$.

Proof. $\implies$: for any $\epsilon > 0$, continuity at x implies that there exists $\delta > 0$ such that $|g(x') - g(x)| < \epsilon$ whenever $d(x', x) < \delta$. Since x is not isolated there exists some $y \in B(x, \delta)$. It follows that, for all $\tilde{a} \in A$, if $y \le \tilde{a}$ then $\tilde{a} \in B(x, \delta)$ and so $|g(\tilde{a}) - g(a)| < \epsilon$, i.e., $g\big|_A \to g(x)$ as a net.
$\impliedby$: let $\{x_n\} \subset X$ be a sequence converging to x. We want to show that $g(x_n) \to g(x)$. WLOG assume $\{x_n\} \subset X \setminus \{x\}$, as this claim is trivial when g acts on x itself. Then for any $\epsilon > 0$ there exists $a_0 \in A$ such that $|g(\tilde{a}) - g(x)| < \epsilon$ whenever $a_0 \le \tilde{a}$. Setting $\delta := d(x, a_0)$, we immediately have the $\epsilon - \delta$ condition of continuity, and the lemma has been proven. ☐

Back to the main proof: since each f_n is continuous at x, by lemma $f_n\big|_A$ converges to $f_n(x)$ as a net. By assumption $f_n \to f$ uniformly so $f_n(x) \to f(x)$. On the other hand, by (b), $f\big|_A$ viewed as a net also converges to $f(x)$. Using the lemma one more time, we conclude that f is continuous at x. ☐

(d) Use part (b) to give an alternate proof that if $[a, b]$ is a closed interval in $\mathbb{R}$ and $f_n : [a, b] \to \mathbb{R}$ are Riemann integrable functions converging uniformly to $f : [a, b] \to \mathbb{R}$, then f is Riemann integrable.

Proof. Let A be the set of tagged partitions (P, T) of $[a, b]$ with $\le$: $(P, T) \le (P'T')$ when P' refines P. By HW1, a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable with integral I if and only if the net $(P, T) \mapsto R(f, P, T)$ converges to I.
Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that, for $n \geqslant N$,

$$|f(\tilde{x}) - f_n(\tilde{x})| < \frac{\epsilon}{b - a} \text{ for all } \tilde{x} \in [a, b].$$

Then it follows that, regardless of choice of (P, T),

$$|R(f, P, T) - R(f_n, P, T)| \leqslant \|f - f_n\|_{\sup}(b - a) < \epsilon.$$

In other words, the nets $(P, T) \mapsto R(f_n, P, T)$ converge uniformly in (P, T) to the net $(P, T) \mapsto R(f, P, T)$. By (a) the limits $R(f_n, P, T)$ form a Cauchy sequence and by completeness of $\mathbb{R}$, these limits converge in $\mathbb{R}$. Finally, by (b), we know the net $(P, T) \mapsto R(f, P, T)$ converges, and this by HW1 implies f is Riemann integrable. ☐