HOMEWORK, WEEK 5

This assignment is due Friday, February 19 in lecture. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

(1) Let $[a, b] \subset \mathbb{R}$ and let $f_n : [a, b] \to \mathbb{R}$ be differentiable functions. Suppose that the derivatives f_n' converge uniformly to a function $g : [a, b] \to \mathbb{R}$ and that for some $x_0 \in [a, b]$, the function values $f_n(x_0)$ converge to some limit L in $\mathbb{R}$ (i.e. (f_n) converges "pointwise somewhere," a very weak condition). Prove that the functions f_n converge uniformly to some function $f : [a, b] \to \mathbb{R}$.

Hint: It suffices to show the sequence of functions (f_n) is uniformly Cauchy; for this, try to relate $|f_n(x) - f_m(x)|$ to $|f_n(x_0) - f_m(x_0)|$ which can be made small for large n, m. You'll have to deal with terms like $|f_n(x) - f_n(x_0)|$; can you show these are small for large n, uniformly in x, using the mean value theorem?

Remark. As we'll discuss in class, this result lets us weaken the hypotheses of Theorem 9 in Section 4.1 of Pugh; in particular, the pointwise limit of a sequence of differentiable functions is differentiable provided that the sequence of derivatives converges uniformly (you don't need to know a priori that convergence of the functions themselves is uniform, only convergence of their derivatives).

(2) Let $f_n : [-1, 1] \to \mathbb{R}$ be defined by $f_n(x) = \sqrt{x^2 + 1/n}$, and let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = |x|$. Prove that the functions f_n converge uniformly to f.

Hint: Can you apply last week's homework?

Remark. As mentioned in the book, this example shows that a uniform limit of differentiable functions may not be differentiable (to show f_n is differentiable or even smooth, use the quotient rule repeatedly). In fact, it's possible to for the uniform limit of smooth functions to be *nowhere* differentiable (an example was given by Weierstrass in 1872 and a related example is discussed in Section 4.7 of Pugh; historically it was very surprising that a continuous nowhere-differentiable function could exist at all).

(3)

Definition. Let (X, d) and (Y, d') be metric spaces and let $f_n : X \to Y$ be a sequence of functions. Let $f : X \to Y$ be another function.

- We say $(f_n)_{n=1}^{\infty}$ *converges compactly* to f if for any compact subset $K \subset X$, the restrictions of f_n to K converge uniformly to the restriction of f to K.
- We say $(f_n)_{n=1}^{\infty}$ *converges locally uniformly* to f if for any $x \in X$, there exists an open neighborhood U of x such that the restrictions of f_n to U converge uniformly to the restriction of f to U.

Show that if (X, d) is *locally compact*, i.e. for all $x \in X$ there exist $x \in U \subset K \subset X$ with U open and K compact, and $f_n, f : X \to Y$ are functions, then the functions f_n converge compactly to f if and only if they converge locally uniformly to f.

Hint: Local compactness should give you (compact convergence $\implies$ local uniform convergence); for the other direction, try using the open-covers characterization of compactness.

Remark. Local uniform convergence is especially important for holomorphic functions; note that any open subset of $\mathbb{C}$ (or $\mathbb{R}$, or $\mathbb{R}^n$) is locally compact. If $f_n : U \to \mathbb{C}$ is a sequence of holomorphic functions converging locally uniformly to $f : U \to \mathbb{C}$, then f is also holomorphic and the derivatives f_n' converge locally uniformly to f'. This is a remarkable fact; in particular, unlike for real-differentiable functions, the uniform limit of holomorphic ("complex-differentiable") functions is always holomorphic ("complex-differentiable"). One can prove this fact using the Cauchy integral formula in complex analysis, differentiation under the integral sign (Theorem 14 in Section 5.2 of Pugh), and a variation of "uniform convergence preserves integrals."

Given this fact, one gets an alternate proof that (complex and thus real) convergent power series are infinitely differentiable and can be differentiated term-by-term in their disk of convergence. We'll prove the result about term-by-term differentiation for real power series using the computation of the radius of convergence for the differentiated series from HW 3, plus our theorem on uniform convergence of the sequence of derivatives. Our proof could also be adapted to the complex setting with a bit more work, although most complex analysis books seem to do the radius-of-convergence computation like us and then do the rest from scratch. In any of the above approaches, the Weierstrass M-test is crucial in showing that power series converge locally uniformly (i.e. converge compactly) on their disk or interval of convergence.

(4) For $[a, b] \subset \mathbb{R}$, consider the vector space $C^0([a, b], \mathbb{R})$ (we could take functions into $\mathbb{C}$ instead and have a complex vector space). We have a norm $\| \cdot \|_{\sup}$ on $C^0([a, b], \mathbb{R})$ making it into a Banach space. In this problem we will consider another norm on $C^0([a, b], \mathbb{R})$ that does not give a complete metric.

Definition. For $f \in C^0([a, b], \mathbb{R})$, define $\|f\|_1 := \int_a^b |f(x)|dx$. The norm axioms hold for $\| \cdot \|_1$ (make sure you check them to your own satisfaction; the requirement that if $\|v\| = 0$ then $v = 0$ holds since a continuous f with $\int_a^b |f(x)|dx = 0$ must be zero everywhere).

(a) Show that if $f_n, f \in C^0([a, b], \mathbb{R})$ and f_n converges to f in $\| \cdot \|_{\sup}$, then f_n converges to f in $\| \cdot \|_1$.
(b) Give an example of $f_n, f \in C^0([0, 1], \mathbb{R})$ such that f_n converges to f in $\| \cdot \|_1$ but not in $\| \cdot \|_{\sup}$.
(c) Show that $(C^0([0, 1], \mathbb{R}), \| \cdot \|_1)$ is not complete (i.e. not a Banach space).

Hint: For the first statement, you can apply a result from class, although it's a bit overpowered here since we already know continuous functions are Riemann integrable. Alternatively, you can argue directly that $\|g\|_1 \leq (b-a)\|g\|_{\sup}$ for all $g \in C^0([a, b], \mathbb{R})$, and use this fact to finish the proof (this argument basically repeats the second part of the corresponding proof in class, after integrability has been proven).

For the second statement, try to find continuous functions f_n on $[0,1]$ such that $\int_0^1 |f_n(x)|dx$ converges to zero as $n \to \infty$, but such that f_n do not converge uniformly to the zero function. You can adapt an example that has already been discussed in Section 4.1 if you want.

For the third statement, you can use functions f_n which are zero on most of $[0,1/2]$, then increase steeply to arrive at $f_n(1/2) = 1$ and have $f_n(x) = 1$ for $x \in [1/2,1]$. As n increases, the functions should wait longer (closer to $1/2$) to start increasing from zero to one, and their steepness should be greater. Show that your functions are Cauchy in d_1. Assume they converge in d_1 to a continuous function $f : [0,1] \to \mathbb{R}$, and try to derive a contradiction: show that the restrictions of f_n to $[0,1/2]$ converge in $(C^0([0,1/2],\mathbb{R}), d_1)$ to both the zero function and to $f|_{[0,1/2]}$. By uniqueness of limits in the metric space $(C^0([0,1/2],\mathbb{R}), d_1)$, it follows that $f|_{[0,1/2]}$ is the zero function. Show that $f|_{[1/2,1]} = 1$ similarly and derive a contradiction.

Remark. One could ask if dropping the continuity requirement and considering $\|\cdot\|_1$ on the vector space of Riemann integrable functions $\mathfrak{R}[a,b]$ fixes the completeness problem. The first issue is that $\|\cdot\|_1$ is not a norm on $\mathfrak{R}[a,b]$: it is possible to have $\|f\|_1 = 0$ without $f = 0$ (continuity prevented this from happening above). To fix this issue, you could define an equivalence relation $f \sim g$ if $\int_a^b |f(x) - g(x)|dx = 0$ and let $\mathfrak{R}^1[a,b] = \mathfrak{R}[a,b]/\sim$; one can check that $\mathfrak{R}^1[a,b]$ is a vector space and $\|\cdot\|_1$ is a (well-defined) norm on $\mathfrak{R}^1[a,b]$.

In fact, $(\mathfrak{R}^1[a,b], \|\cdot\|_1)$ is also not complete, i.e. not a Banach space. However, if you perform the same construction with Lebesgue integrals (as covered in Math 525a), you get a Banach space known as $L^1([a,b])$ (this is a main advantage of the Lebesgue integral). In fact, there are Banach spaces $L^p([a,b])$ for $1 \leq p < \infty$, defined similarly with $\|f\|_p = (\int_a^b |f(x)|^p dx)^{1/p}$ (and consisting of functions f with $|f|^p$ Lebesgue integrable, modulo equivalence). In all cases the equivalence relation is the same as almost-everywhere equality, and the L^p spaces can be generalized to $L^p(X,\Sigma,\mu)$ for an arbitrary measure space (X,Σ,μ) (see 525a for measure spaces as well as Lebesgue integrals).

For $(X,\Sigma,\mu) = \mathbb{N}$ with the counting measure, one recovers ℓ^p spaces of sequences, and for $(X,\Sigma,\mu) = \{1,\ldots,n\}$ with the counting measure, one recovers ℓ^p norms on $\mathbb{R}^n$, including the familiar Euclidean norm ℓ^2. In general, L^p is only a Hilbert space when $p = 2$; for this reason, L^2 is especially important (e.g. $L^2(\mathbb{R}^3;\mathbb{C})$ is the quantum Hilbert space for a particle moving in $\mathbb{R}^3$).

For $p = \infty$, we have studied (generalizations of) $C_b(X,\mathbb{R})$ when X is a set. If we have a measure-space structure on X, we can study a related space $L^\infty(X,\Sigma,\mu;\mathbb{R}) =: L^\infty(X)$. Its elements are "essentially bounded" measurable functions modulo almost-everywhere inequality, rather than just plain bounded functions, and the norm $\|\cdot\|_\infty$ is the "essential supremum" (a measure-zero-ignoring version of the sup norm $\|\cdot\|_{\sup}$ we have been considering). For $X = [a,b]$ with Lebesgue measure, continuous functions that are equal almost everywhere are equal everywhere, and the essential supremum on continuous functions coincides with the sup norm we have defined.

Thus, you showed above that on the subspace of continuous functions inside $L^\infty([a,b])$, convergence in L^∞ implies convergence in L^1. In general, if (X, Σ, μ) does not have sets of finite but arbitrarily large measure (e.g. $X = [a,b]$), then for $p \leq q \in [1,\infty]$, $L^q(X) \subset L^p(X)$ and $\|\cdot\|_p \leq C\|\cdot\|_q$ for some constant C. If (X, Σ, μ) does not have sets of nonzero but arbitrarily small measure (e.g. $X = \mathbb{N}$), then for $p \leq q \in [1,\infty]$, $L^p(X) \subset L^q(X)$ and $\|\cdot\|_q \leq C\|\cdot\|_p$ for some constant C.