MATH 425b Homework 5

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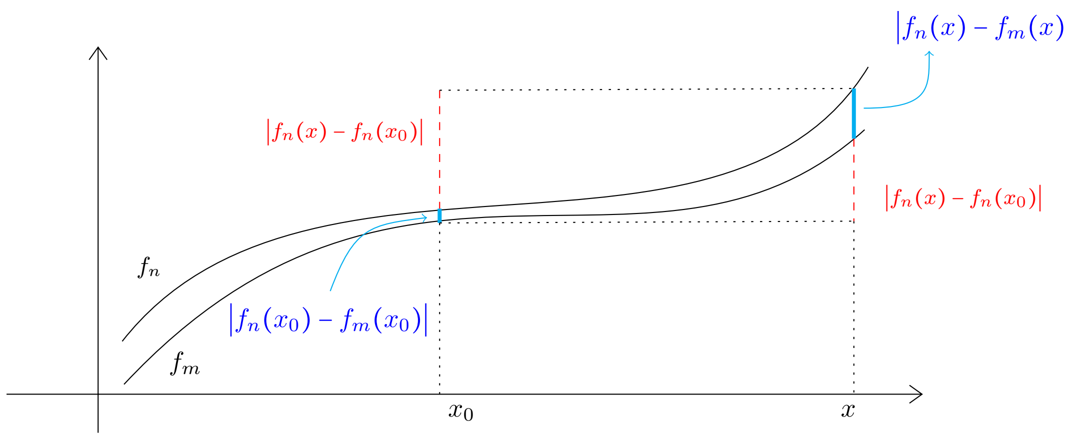
Problem 1

Let $[a,b] \subset \mathbb{R}$ and let $f_n : [a,b] \to \mathbb{R}$ be differentiable functions. Suppose that the derivatives f_n' converge uniformly to a function $g : [a,b] \to \mathbb{R}$ and that for some $x_0 \in [a,b]$, the function values $f_n(x_0)$ converge to some limit L in $\mathbb{R}$ (i.e. $\{f_n\}$ converges "pointwise somewhere," a very weak condition). Prove that the functions f_n converge uniformly to some function $f : [a,b] \to \mathbb{R}$.

Proof. Let $\epsilon > 0$ be given. Since $\mathbb{R}$ is complete, it suffices to show that $\{f_n\}$ is uniformly Cauchy, i.e., for sufficiently large m, n, $|f_m(x) - f_n(x)| < \epsilon$ for all $x \in [a,b]$. First notice that, by triangle inequality,

$$\begin{aligned} |f_m(x) - f_n(x)| &= |f_m(x) - f_m(x_0) + f_m(x_0) - f_n(x_0) + f_n(x_0) - f_n(x)| \\ &\leqslant |f_m(x_0) - f_n(x_0)| + \big|[f_m(x) - f_m(x_0)] - [f_n(x_0) - f_n(x)]\big| \\ &= |f_m(x_0) - f_n(x_0)| + |(f_m - f_n)(x) - (f_m - f_n)(x_0)|. \end{aligned}$$



The first term can be easily bounded by $\epsilon/2$ since pointwise convergence at x_0 implies Cauchy-ness there. Let N_1 be the lower bound of m, n in that corresponding $\epsilon - N$ condition.

To bound the second term, notice that since f_m, f_n are differentiable, so is $f_m - f_n$. By mean value theorem there exists $\xi \in [a,b]$ such that

$$(f_m - f_n)(x) - (f_m - f_n)(x_0) = (b-a)(f_m - f_n)'(\xi).$$

On the other hand, notice that $(f_m - f_n)' = f_m' - f_n'$. The uniform convergence of the derivatives also imply uniform Cauchy-ness, and so there exists $N_2 \in \mathbb{N}$ such that

$$(f_m - f_n)'(\tilde{x}) = (f_m)'(\tilde{x}) - (f_n)'(\tilde{x}) < \frac{\epsilon}{2(b-a)} \text{ for all } \tilde{x} \in [a,b].$$

Taking $N := \max\{N_1, N_2\}$, we have

$$|f_m(x) - f_n(x)| \leqslant |f_m(x_0) - f_n(x_0)| + (b-a)(f_m - f_n)'(\xi) < \frac{\epsilon}{2} + \frac{\epsilon}{2(b-a)}(b-a) = \epsilon,$$

and the claim follows. □

Problem 2

Let $f_n : [-1,1] \to \mathbb{R}$ be defined by $f_n(x) = \sqrt{x^2 + 1/n}$ and let $f : [-1,1] \to \mathbb{R}$ be defined by $f(x) = |x|$. Prove that the functions f_n converge uniformly to f.

Proof. Since $(a+b)^2 = a^2 + 2ab + b^2$, one can easily show that $\sqrt{a^2 + b^2} \leqslant a + b$ for nonnegative a and b. Then,

$$f_n(x) - f(x) \leqslant \begin{Bmatrix} \sqrt{x^2 + (1/\sqrt{n})^2} - |x| & x \geqslant 0 \\ \sqrt{x^2 + (1/\sqrt{n})^2} - |x| & x < 0 \end{Bmatrix} = \frac{1}{\sqrt{n}},$$

and uniform convergence follows by taking $N > (1/\epsilon)^2$ for any $\epsilon > 0$ given. □

Alternate Proof. This claim directly follows from Dini's theorem from HW4. □

Definition

Let (X, d) and (Y, d') be metric spaces and let $f_n : X \to Y$ be a sequence of functions. Let $f : X \to Y$ be another function.

(1) We say (X, d) is **locally compact** if for all $x \in X$ there exists open neighborhood $U \subset K \subset X$ with K open.

(2) We say $\{f_n\}_{n \geqslant 1}$ **converges compactly** to f if, for any compact subset $K \subset X$, $f_n\big|_K$ converges uniformly to $f\big|_K$.

(3) We say $\{f_n\}_{n \geqslant 1}$ **converges locally uniformly** to f if, for any $x \in X$, there exists an open neighborhood U of x such that $f_n\big|_U \to f\big|_U$ uniformly.

Problem 3

Show that if (X, d) is locally compact, and $f_n, f : X \to Y$ are functions, then $f_n \to f$ compactly if and only if $f_n \to f$ locally uniformly.

Proof. $\Longrightarrow$ is trivial: $f_n\big|_K \to f\big|_K$ immediately gives us $f_n\big|_U \to f\big|_U$, since we know $U \subset K$.
For $\Longleftarrow$, let $K \subset X$ be an arbitrary compact set. By the assumption that (X, d) is locally compact, for each $x \in K$ there exists some open neighborhood U containing x. Doing so, we obtain a covering of K, and by its compactness there exists a finite subcovering

$$\bigcup_{i=1}^{n} U_i \supset K.$$

Let $\epsilon > 0$ be given. Using the assumption that $f_n \to f$ locally uniformly, there exists $N_1 \in \mathbb{N}$ such that $d'(f_m(x), f(x)) < \epsilon$ for all $x \in U_1$, whenever $m \geqslant N_1$. We can repeat this process for all scraps in the open covering, i.e., there exists N_i such that $d'(f_m(x), f(x)) < \epsilon$ for all $x \in U_i$ whenever $m \geqslant N_i$. Since this covering is finite, we are allowed to define

$$N := \max_{1\leqslant i\leqslant n} N_i.$$

It follows that, if $m \geqslant N$ then $d'(f_m(x), f(x)) < \epsilon$ for all $x \in \bigcup_{i=1}^{n} U_i$ and in particular for all $x \in K$. This shows the uniform convergence on K, and the claim follows. □

Problem 4

For $[a,b] \subset \mathbb{R}$, consider the vector space $C^0([a,b],\mathbb{R})$.

(1) Show that if $f_n, f \in C^0([a,b],\mathbb{R})$ and $f_n \to f$ in $\|\cdot\|_{\sup}$, then f_n converges to f in $\|\cdot\|_1$.

(2) Give an example of $f_n, f \in C^0([0,1],\mathbb{R})$ such that f_n converges to f in $\|\cdot\|_1$ but not in $\|\cdot\|_{\sup}$.

(3) Show that $(C^0[0,1],\mathbb{R},\|\cdot\|_1)$ is not Banach.

Proof(s).

(1) Let $\epsilon > 0$ be given. By the uniform convergence w.r.t. $\|\cdot\|_{\sup}$, there exists $N \in \mathbb{N}$ such that $\|f_n - f\|_{\sup} < \epsilon/(b-a)$ whenever $n \geqslant N$. This means $|f_n(x) - f(x)| < \epsilon/(b-a)$ for all $x \in [a,b]$. Therefore,

$$\|f_n - f\|_1 = \int_a^b |f_n(\tilde{x}) - f(\tilde{x})| \, \mathrm{d}\tilde{x} < \frac{\epsilon}{b-a}(b-a) = \epsilon,$$

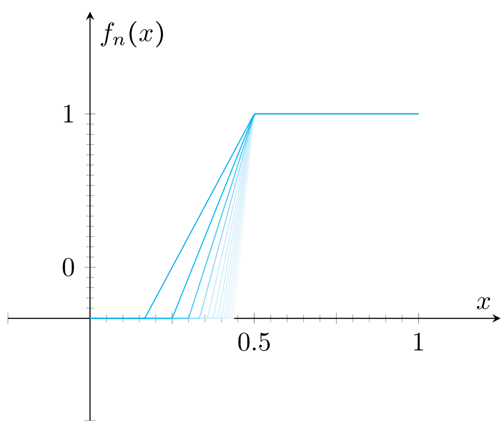
i.e., $f_n \to f$ w.r.t. $\|\cdot\|_1$ as well.

(2) Consider $\{f_n\}_{n\geqslant 1} \subset C^0([0,1],\mathbb{R})$ defined by $f_n := x^n$. Then $\|f_n\|_1 = \int_0^1 \tilde{x}^n \, \mathrm{d}\tilde{x} = \left.\frac{\tilde{x}^{n+1}}{n+1}\right|_{\tilde{x}=0}^{1} = \frac{1}{n+1}$. Clearly as $n \to \infty$, $\|f_n\|_1 \to 0 = \|f\|_1$ where $f \equiv 0$.

On the other hand, $\|f_n\|_{\sup} = f_n(1) = 1$ for all n, whereas $\|f\|_{\sup} = 0$.

(3) Consider the sequence of functions $\{f_n\}$ where $f_n(x) \equiv 0$ for $x \in [0, 1/2 - 1/n]$, $f_n(x) \equiv 1$ for $x \in [1/2, 1]$, and $f_n(x)$ increases with slope n over the interval $(1/2 - 1/n, 1/2)$. See the figure below.

$$f_n(x) = \begin{cases} 0, & x \in [0, 0.5 - 1/n], \\ nx - \dfrac{n-2}{2} & x \in (0.5 - 1/n, 0.5), \\ 1 & x \in [0.5, 1]. \end{cases}$$



Now define a piecewise function $f : [0,1] \to \mathbb{R}$ with $f(x) \equiv 0$ for $x \in [0, 0.5)$ and $f(x) \equiv 1$ for $x \in [0.5, 1]$. We'll show that $f_n \to f$ with respect to $\|\cdot\|_1$:

$$\|f - f_n\|_1 = \int_0^1 f(\tilde{x}) - f_n(\tilde{x}) \, \mathrm{d}\tilde{x} = \int_{0.5-1/n}^{0.5} n\tilde{x} - \frac{n-2}{2} \, \mathrm{d}\tilde{x} = \frac{1}{2n}$$

which converges to 0 as $n \to \infty$. However, $f \notin C^0([0,1],\mathbb{R})$ and so $(C^0([0,1]), \|\cdot\|_1)$ is not Banach.

□