HOMEWORK, WEEK 6

This assignment is due Monday, March 1 in lecture. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

(1) Let $\{f_1, \ldots, f_k\}$ be a finite set of uniformly continuous functions from X to Y, where X and Y are two metric spaces. Prove that $\{f_1, \ldots, f_k\}$ is equicontinuous.

(2) Let $\{f_\alpha\}_{\alpha \in A}$ be an equicontinuous set of functions from X to Y, and let g be a uniformly continuous function from Y to Z (where X, Y, and Z are metric spaces). Prove that the set of functions $\{g \circ f_\alpha\}_{\alpha \in A}$ is equicontinuous.

Hint: Given $\varepsilon > 0$, use uniform continuity of g to choose δ appropriately, then use this δ as "ε" in the definition of equicontinuity of the set $\{f_\alpha\}$.

(3) This problem is based on the introduction to André Weil's book *Elliptic functions according to Eisenstein and Kronecker*, where he suggests that one reason for the historical underappreciation of Eisenstein's work was that the notion of uniform convergence had not been developed in Eisenstein's time and to later authors like Weierstrass, Eisenstein's work may have seemed non-rigorous (for example, he implicitly assumes that the series he introduces can be differentiated term-by-term). In his introduction, Weil outlines why there is no issue with the types of series considered by Eisenstein; we will see how the theorem we proved on uniform convergence and derivatives gives a straightforward argument for this fact.

(a) The examples considered in Weil's introduction are the doubly-infinite series $\sum_{\mu=-\infty}^{\infty} \frac{1}{(z-\mu)^n}$ for $n \geq 1$, which we treat as singly-infinite series by taking symmetric partial sums (when $n = 1$ this is necessary for convergence of the series). Correspondingly, consider the series

$$\sum_{\mu=M}^{\infty} \left(\frac{1}{(z-\mu)^n} + \frac{1}{(z+\mu)^n} \right)$$

of functions of a complex variable z (when discussing convergence, we will feel free to discard finitely many terms at the beginning of the series and start from $\mu = M$).

Show that for any $z \in \mathbb{C}$ with $|z| < M$, the above series converges absolutely (the restriction $|z| < M$ is so that there's no question about whether some term of the series involves division by zero).

Hint: For $n > 1$, try comparing with a convergent p-series. For $n = 1$, rewrite the series as $\sum_{\mu=M}^{\infty} \frac{2z}{z^2-\mu^2}$ so you can again compare with a convergent p-series.

(b) Now, for $z \in \mathbb{C} \setminus \mathbb{Z}$, let

$$f(z) = \frac{1}{z} + \sum_{\mu=1}^{\infty} \left(\frac{1}{(z-\mu)^n} + \frac{1}{(z+\mu)^n} \right).$$

Prove, using our theorem on uniform convergence and derivatives, that f is holomorphic (complex differentiable) on $\mathbb{C} \setminus \mathbb{Z}$ and that its derivative $f'(z)$ is given by differentiating the above series term-by-term.

Hint: Try discarding finitely many terms (each term is holomorphic) and applying the theorem on uniform convergence and derivatives to the remaining series, in a neighborhood of any given point $z_0 \in \mathbb{C} \setminus \mathbb{Z}$ (this theorem holds in the complex setting too, as we'll discuss briefly in Monday's lecture). When showing the term-by-term differentiated series converges uniformly in a neighborhood of z_0, it might help to revisit the first part of the problem where the hint suggested comparison with a p-series. Can you use the Weierstrass M-test to show this convergence is uniform in z, for z in a neighborhood of z_0? Remember that n is constant!

Remark. In fact, for $n = 1$ we have $f(z) = \pi \cot(\pi z)$; see Weil's book for much more on these functions and their generalizations, which have many interesting relationships with number theory.

Remark. If we had access to the theorems of complex analysis, then we wouldn't need to show local uniform convergence of the series of derivatives in order to get that the limit function is holomorphic; local uniform convergence of the original series itself would be enough (since the terms are holomorphic). It turns out that the proof is basically the same either way in this case, though. In general, the main advantage of our theorem on uniform convergence and derivatives is that it holds in the real-variables case where one may have differentiability without analyticity, but for the example at hand complex techniques are natural since the terms of the series can be naturally viewed as holomorphic functions.

Remark. Weil's argument uses neither technology from complex analysis nor our theorem on uniform convergence and derivatives (which also counts as technology from analysis in some sense). Instead, he uses the binomial series to derive an expression for $f(x + y)$ as a series whose terms are themselves series obtained by repeatedly differentiating the series for f term-by-term. From here, one can directly compute the limit of $\frac{f(x+y)-f(x)}{y}$ as y goes to zero, yielding the first term-by-term derivative of the series for f.