

Proof of (b). Consider the sequence of functions $\{f_k\}$ defined by

$$f_k(z) = \frac{1}{z} + \sum_{\mu=1}^k \left(\frac{1}{(z-\mu)^n} + \frac{1}{(z+\mu)^n} \right).$$

By (a), we know that f_k converges to f pointwise. It remains to show that f'_k converges uniformly to some g ; then f is holomorphic with $f' = g$. First thing to notice is that each f_k is holomorphic with

$$f'_k(z) = -\frac{1}{z^2} - \sum_{\mu=1}^k \left(\frac{n}{(z-\mu)^{n+1}} + \frac{n}{(z+\mu)^{n+1}} \right).$$

Let $z_0 \in \mathbb{C} \setminus \mathbb{Z}$ be given. Consider a open neighborhood $z_0 \in \Omega \subset \mathbb{C} \setminus \mathbb{Z}$. Let $2M \in \mathbb{N}$ be such that $|z| < 2M$ for all $z \in \Omega$. Then by the previous part we know

$$\begin{aligned} \left| -\frac{1}{z^2} - \sum_{\mu=2M}^k \left(\frac{n}{(z-\mu)^{n+1}} + \frac{n}{(z+\mu)^{n+1}} \right) \right| &\leq \frac{1}{|z|^2} + \sum_{\mu=2M}^k \left(\frac{n}{|z-\mu|^{n+1}} + \frac{n}{|z+\mu|^{n+1}} \right) \\ &< \frac{1}{|z|^2} + \sum_{\mu=2M}^k \frac{2n}{(\mu/2)^{n+1}} \\ &< \frac{1}{4M^2} + \sum_{\mu=2M}^k \frac{2n}{(\mu/2)^{n+1}} \end{aligned}$$

Since $n+1 > 1$, $\frac{1}{4M^2} + \sum_{\mu=2M}^{\infty} \frac{2n}{(\mu/2)^{n+1}}$ converges. Thus $\|f_k\|$ is bounded by a convergent series, and by Weierstraß

M-test, the series f'_k converges uniformly on Ω . Therefore

$$f'(z) = -\frac{1}{z^2} + \sum_{\mu=1}^{\infty} \left(\frac{n}{(z-\mu)^{n+1}} + \frac{n}{(z+\mu)^{n+1}} \right),$$

and its differentiability implies that it is holomorphic. □