MATH 425B HW6

Qilin Ye

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Problem 1

Let $\{f_1, \ldots, f_k\}$ be a finite set of uniformly continuous functions from X to Y where X and Y are two metric spaces. Prove that $\{f_1, \ldots, f_k\}$ is equicontinuous.

Proof. Let $\epsilon > 0$ be given. Since f_1 is uniformly continuous, there exists δ_1 such that

$$x_1, x_2 \in X, d_X(x_1, x_2) < \delta_1 \implies d_Y(f_1(x_1), f_1(x_2)) < \epsilon.$$

A similar argument can be applied to each f_n in the set, i.e., there exists δ_n such that

$$x_1, x_2 \in X, d_X(x_1, x_2) < \delta_n \implies d_Y(f_n(x_1), f_n(x_2)) < \epsilon.$$

Now we define $\delta := \min\{\delta_1, \ldots, \delta_k\}$. This is well-defined because there are only finitely many δ's. It follows that, for all $f_n \in \{f_1, \ldots, f_k\}$ and $x_1, x_2 \in X$,

$$d_X(x_1, x_2) < \delta \implies d_X(x_1, x_2) < \delta_n \implies d_Y(f_n(x_1), f_n(x_2)) < \epsilon,$$

hence the equicontinuity. □

Problem 2

Let $\{f_\alpha\}_{\alpha\in A}$ be an equicontinuous set of functions $X \to Y$ and let g be a uniformly continuous function from $Y \to Z$. Prove that $\{g \circ f_\alpha\}_{\alpha\in A}$ is equicontinuous.

Proof. Let $\epsilon(z) > 0$ be given. By the equicontinuity of g, there exists $\epsilon(y) > 0$ (this should be the "δ" corresponding to the previous $\epsilon(z)$) such that

$$y_1, y_2 \in Y, d_Y(y_1, y_2) < \epsilon(y) \implies d_Z(g(y_1), g(y_2)) < \epsilon(z).$$

Now since $\{f_\alpha\}_{\alpha\in A}$ is equicontinuous, given this $\epsilon(y)$ there exists a $\delta(x) > 0$ such that, for all $n \in A$,

$$\begin{aligned} x_1, x_2 \in X, d_X(x_1, x_2) < \delta(x) &\implies d_Y(f_n(x_1), f_n(x_2)) < \epsilon(y) \\ &\implies d_Z\left[g(f_n(x_1)), g(f_n(x_2))\right] < \epsilon(z) \end{aligned}$$

and the claim follows. □

Problem 3

(a) Consider the series

$$\sum_{\mu=M}^{\infty}\left(\frac{1}{(z-\mu)^n}+\frac{1}{(z+\mu)^n}\right)$$

of functions of a complex variable z. Show that for any $z \in \mathbb{C}$ with $|z| < M$, the above series converges absolutely for $n \geqslant 1$.

(b) For $z \in \mathbb{C} \setminus \mathbb{Z}$, let

$$f(z) = \frac{1}{z} + \sum_{\mu=1}^{\infty}\left(\frac{1}{(z-\mu)^n}+\frac{1}{(z+\mu)^n}\right).$$

Prove, using our theorem on uniform convergence and derivatives, that f is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$ and that its derivative $f'(z)$ is given by differentiating the above series term by term.

Proof of (a). If $p > 1$, notice that by triangle inequality

$$|-\mu| \leqslant |z-\mu| + |-z| \implies |z-\mu| \geqslant |\mu| - |z|$$
$$|\mu| \leqslant |z+\mu| + |-z| \implies |z+\mu| \geqslant |\mu| - |z|.$$

Therefore

$$\begin{aligned}
\sum_{\mu=M}^{\infty}\left|\frac{1}{(z-\mu)^n}+\frac{1}{(z+\mu)^n}\right| &\leqslant \sum_{\mu=M}^{\infty}\left(\frac{1}{|z-\mu|^n}+\frac{1}{|z+\mu|^n}\right) \\
&\leqslant \sum_{\mu=M}^{2M-1}\left(\frac{1}{|z-\mu|^n}+\frac{1}{|z+\mu|^n}\right) + \sum_{\mu=2M}^{\infty}\left(\frac{1}{(|\mu|-|z|)^n}+\frac{1}{(|\mu|-|z|)^n}\right) \\
&< \text{finite} + \sum_{\mu=2M}^{\infty}\frac{2}{(\mu/2)^n} \qquad (\text{since } \mu \geqslant 2M \implies |z| < |\mu|/2) \\
&< \infty.
\end{aligned}$$

Similarly, when $p = 1$ we have $\frac{1}{z-\mu}+\frac{1}{z+\mu} = \frac{2z}{z^2-\mu^2}$, and by triangle inequality

$$|-\mu^2| \leqslant |z^2-\mu^2| + |-z^2| \implies |z^2-\mu^2| \geqslant |\mu|^2 - |z|^2.$$

It follows that

$$\begin{aligned}
\sum_{\mu=M}^{\infty}\left|\frac{2z}{z^2-\mu^2}\right| &= \sum_{\mu=M}^{2M-1}\left|\frac{2z}{z^2-\mu^2}\right| + \sum_{\mu=2M}^{\infty}\left|\frac{2z}{z^2-\mu^2}\right| \\
&\leqslant \text{finite} + \sum_{\mu=2M}^{\infty}\frac{|2z|}{|\mu|^2-|z|^2} \\
&< \text{finite} + \sum_{\mu=2M}^{\infty}\frac{|8z/3|}{\mu^2} \qquad (\text{since } \mu \geqslant 2M \implies |z|^2 < \mu^2/4) \\
&< \infty.
\end{aligned}$$

□

Proof of (b). Consider the sequence of functions $\{f_k\}$ defined by

$$f_k(z) = \frac{1}{z} + \sum_{\mu=1}^{k} \left(\frac{1}{(z-\mu)^n} + \frac{1}{(z+\mu)^n} \right).$$

By (a), we know that f_k converges to f pointwise. It remains to show that f_k' converges uniformly to some g; then f is holomorphic with $f' = g$. First thing to notice is that each f_k is holomorphic with

$$f_k'(z) = -\frac{1}{z^2} - \sum_{\mu=1}^{k} \left(\frac{n}{(z-\mu)^{n+1}} + \frac{n}{(z+\mu)^{n+1}} \right).$$

Let $z_0 \in \mathbb{C} \setminus \mathbb{Z}$ be given. Consider a open neighborhood $z_0 \in \Omega \subset \mathbb{C} \setminus \mathbb{Z}$. Let $2M \in \mathbb{N}$ be such that $|z| < 2M$ for all $z \in \Omega$. Then by the previous part we know

$$\begin{aligned}
\left| -\frac{1}{z^2} - \sum_{\mu=2M}^{k} \left(\frac{n}{(z-\mu)^{n+1}} + \frac{n}{(z+\mu)^{n+1}} \right) \right| &\leqslant \frac{1}{|z|^2} + \sum_{\mu=2M}^{k} \left(\frac{n}{|z-\mu|^{n+1}} + \frac{n}{|z+\mu|^{n+1}} \right) \\
&< \frac{1}{|z|^2} + \sum_{\mu=2M}^{k} \frac{2n}{(\mu/2)^{n+1}} \\
&< \frac{1}{4M^2} + \sum_{\mu=2M}^{k} \frac{2n}{(\mu/2)^{n+1}}
\end{aligned}$$

Since $n+1 > 1$, $\frac{1}{4M^2} + \sum_{\mu=2M}^{\infty} \frac{2n}{(\mu/2)^{n+1}}$ converges. Notice the summation above starts from $\mu = 2M$ so that we can directly compare it with a p-series, but adding the ignored $2M-1$ terms back won't affect the convergence. Thus $\|f_k'\|$ is bounded by a convergent series, and by Weierstraß M-test, the series f_k' converges uniformly on Ω. Therefore

$$f'(z) = g = -\frac{1}{z^2} + \sum_{\mu=1}^{\infty} \left(\frac{n}{(z-\mu)^{n+1}} + \frac{n}{(z+\mu)^{n+1}} \right),$$

and its differentiability implies that it is holomorphic. □