HOMEWORK, WEEK 7

This assignment is due Monday, March 8 in lecture. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

(1) (Pugh Exercise 4.19, countable case): Let X be a compact metric space and let $A = \{a_n\}_{n \geq 1}$ be a countable dense subset of X. For any $\delta > 0$, show that there exists M such that for any $x \in X$, we have $d(x, a_i) < \delta$ for some i with $1 \leq i \leq M$. (This problem supplies an important step in the proof of the Arzelá–Ascoli propagation theorem.)

Hint: Can you find a way to apply Dini's theorem?

(2) Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{\sin^2(n^n x)}{\sqrt{n+2}}$ for $n \geq 1$. Prove that the sequence $(f_n)_{n=1}^{\infty}$ is equicontinuous.

Hint: Given $\varepsilon > 0$, choose N such that $\frac{1}{\sqrt{N+2}} < \varepsilon/2$. Use that the functions $\{f_1, \ldots, f_{N-1}\}$ form an equicontinuous set (from last week's homework).

(3) Pugh, Exercise 4.22.

Hint: Can the example from the previous problem help you out? Remember to restrict the domain to a compact subinterval, to answer the question as stated in the problem.

(4) In next week's homework we will explore an application of the Stone–Weierstrass theorem to Fourier series; we need a few preliminary constructions first.

Remark. Fourier series and (especially) Fourier transforms are most naturally treated using the Lebesgue integral, but one can say many interesting things about them without setting up this technology, and this is good to do given the applicability of Fourier analysis in pure and applied math, science and engineering, etc. For an elementary perspective on Fourier analysis, see Stein and Shakarchi *Fourier Analysis, an Introduction* which should be readable given your background from this course. More advanced treatments can be found in most graduate-level analysis books.

(a) Let S^1 denote the unit circle in $\mathbb{C} \cong \mathbb{R}^2$, i.e. the set of $x + iy \in \mathbb{C}$ such that $x^2 + y^2 = 1$. Note that S^1 is a compact metric space (it is a closed and bounded subset of $\mathbb{R}^2$).

Let $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ denote the subset of $C^0(\mathbb{R}, \mathbb{C})$ consisting of periodic functions, i.e. functions $f : \mathbb{R} \to \mathbb{C}$ such that $f(\theta + n) = f(\theta)$ for all $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$. Define a map $\Phi : C^0(S^1, \mathbb{C}) \to C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ by

$$\Phi(f)(\theta) = f(e^{2\pi i \theta}).$$

Problem. Prove that Φ is well-defined and bijective.

You may assume basic properties of complex exponentials without proof, e.g. that $e^{zw} = e^z e^w$ for all $z, w \in \mathbb{C}$. You may also use the following fact from complex analysis:

Proposition (Existence and continuity of complex logarithms)**.** For any ray ρ starting at the origin in $\mathbb{C}$, there exists a continuous (even holomorphic) function $\log : \mathbb{C} \setminus \rho \to \mathbb{C}$ such that $e^{\log(z)} = z$ for all $z \in \mathbb{C} \setminus \rho$ (the ray ρ is called a *branch cut*). If the terminology is unfamiliar, a ray is a semi-infinite line with only one endpoint (taken to be the origin here).

Hint: For well-definedness, check that $\Phi(f)$ is continuous and periodic. For continuity, write $\Phi(f)$ as a composition; for periodicity, use properties of exponentials. For injectivity, first show that that any $z = u + iv \in S^1$ is equal to e^w for some $w \in \mathbb{C}$; indeed, you can pick a ray ρ not containing z and then let $w = \log(z)$ where log is defined using the branch cut ρ as above. Then show that w must be purely imaginary using properties of exponentials. It follows that if two functions f, g from S^1 to $\mathbb{C}$ agree at all points $e^{2\pi i\theta}$, then they agree everywhere (since all points in S^1 are of this form).

For surjectivity, given a periodic function F from $\mathbb{R}$ to $\mathbb{C}$, define $f : S^1 \to \mathbb{C}$ by $f(u + iv) = F(\theta)$ for some $\theta \in \mathbb{R}$ with $e^{2\pi i\theta} = u + iv$. You can show that f is well-defined using periodicity of F; you want to show that f is a continuous function from S^1 to $\mathbb{C}$. If you can show that for any $u + iv$ in S^1, there's an open subset W of S^1 such that $f|_W$ (the restriction of f to W) is continuous, then it follows that f is continuous. Again, you can use the above proposition to show that f is continuous on the open subsets $(\mathbb{C} \setminus \rho) \cap S^1$ of S^1 (write it as a composition); any point $u + iv$ is contained in one of these subsets for some ρ. Once you know that f is continuous, then $\Phi(f)$ makes sense, and it follows from the definition of f that $\Phi(f) = F$.

(b)

Problem. Prove that the map Φ defined in the previous problem is linear and preserves the uniform norm.
Hint: Linearity should be very quick, but you should make sure you're checking the right condition (don't try to show it's linear in θ!). To show it preserves the uniform norm, show that for $f, g \in C^0(S^1, \mathbb{C})$, the set of distances $\{|f(z) - g(z)| \mid z \in S^1\}$ is equal to the set of distances $\{|f(e^{2\pi i\theta}) - g(e^{2\pi i\theta})| \mid \theta \in \mathbb{R}\}$.

(c) From the above part, we can now identify $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ and $C^0(S^1, \mathbb{C})$ as normed vector spaces given $\|\cdot\|_{\sup}$; in particular, $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ is complete (i.e. a Banach space) given $\|\cdot\|_{\sup}$.

However, there is another norm on this infinite dimensional vector space that's especially interesting: the L^2 norm. One caveat is that the resulting metric is not complete; in many ways it's more natural to work with the "L^2 space" of Lebesgue square-integrable functions modulo almost-everywhere equality, of which our vector space of continuous functions is a non-closed subspace, but we'll stick with continuous functions here.

By definition, a complex-valued function $f(x) = u(x) + iv(x)$ of a real variable $x \in [a, b]$ is Riemann integrable if its real and imaginary parts $u(x)$ and $v(x)$ are Riemann integrable; in this case we define $\int_a^b f(x)dx = \int_a^b u(x)dx + i\int_a^b v(x)dx$. All results proved about real-valued Riemann integrable functions continue to hold in the complex case, when they make sense (no monotonicity etc.).

Let $f, g \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$. Define

$$\langle f, g\rangle_2 = \int_0^1 \overline{f(x)}g(x)dx,$$

where $\overline{f(x)}$ denotes the complex conjugate of f (this is a choice of convention; sometimes one conjugates g instead).

Problem. Show that $\langle\cdot,\cdot\rangle_2$ is a complex inner product on $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$, i.e. show that the following axioms are satisfied for $\langle\cdot,\cdot\rangle_2$:

- $\langle f, g\rangle = \overline{\langle g, f\rangle}$ for all f, g
- $\langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle$ for all f_1, f_2, g
- $\langle f, cg\rangle = c\langle f, g\rangle$ for all c, f, g
- $\langle f, f\rangle$ is a real number that is ≥ 0 for all f, with equality if and only if $f = 0$.

Hint: All properties are immediate except the statement about equality in the last axiom. For this equality statement, you can use a result from Section 3.2 of Pugh's book.

(d) For infinite-dimensional inner product spaces like $(C^0_{\text{per}}(\mathbb{R}, \mathbb{C}), \langle,\rangle_2)$, the usual notion of basis or orthonormal basis from linear algebra is not always so useful. It's more common to have a linearly independent set that's not literally a spanning set, but which spans in an "infinite" sense: any element of the vector space should be a limit of finite linear combinations of basis elements. Thus, we make the following standard definition, even though it conflicts a bit with the terminology for finite-dimensional vector spaces:

Definition. An *orthonormal basis* for a real or complex inner product space $(V, \langle\cdot,\cdot\rangle)$ is a subset β of V such that:

- For $e, e' \in \beta$ we have $\langle e, e'\rangle = 0$ if $e \neq e'$ and $\langle e, e\rangle = 1$ (i.e. β is an orthonormal set; this condition implies that β is linearly independent)
- The span of β (set of finite linear combinations of elements of β) is dense in V, where the metric on V is defined by $d(v, w) = \|v - w\|$ (where $\|v\| = \sqrt{\langle v, v\rangle}$).

For any $n \in \mathbb{Z}$, we have a periodic function $e_n : \mathbb{R} \to \mathbb{C}$ given by $e_n(\theta) = e^{2\pi in\theta}$. In next week's problem set we will show that the functions e_n form an orthonormal basis for $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$.

Problem. In this problem, just check orthonormality: show that for $n \in \mathbb{Z}$, we have $\langle e_n, e_m\rangle = \delta_{m,n}$, i.e. one if $n = m$ and zero otherwise.

Hint: Using $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, you can show that $\overline{e_n} = e_{-n}$. Thus, you can simplify the expression $\overline{e_n}e_m$ with properties of exponentials and then integrate as usual in calculus.

(e) Now we review some linear algebra. Let $(V, \langle\cdot,\cdot\rangle)$ be a real or complex inner product space and let W be a finite-dimensional subspace of V with an orthonormal basis $\{w_1, \ldots, w_n\}$. For $v \in V$, define the orthogonal projection of v onto W to be

$$\mathrm{proj}_W(v) := \langle w_1, v\rangle w_1 + \cdots + \langle w_n, v\rangle w_n$$

(this looks like it might depend on the orthonormal basis; the next problem implies that it doesn't).

Problem. Show that $\mathrm{proj}_W(v) \in W$ and $v - \mathrm{proj}_W(v)$ is in the orthogonal complement $W^\perp$ of W, i.e. that $\langle w, v - \mathrm{proj}_W(v)\rangle = 0$ for all $w \in W$. Also show that these properties characterize $\mathrm{proj}_W(v)$ uniquely, i.e. if $v = v_1 + v_2 = v_1' + v_2'$ with $v_1, v_1' \in W$ and $v_2, v_2' \in W^\perp$ then $v_1 = v_1'$ and $v_2 = v_2'$.

Hint: to check $v - \mathrm{proj}_W(v)$ is in $W^\perp$, it suffices to check orthogonality of $v - \mathrm{proj}_W(v)$ with each element of an orthonormal basis for W. For the unique characterization, try some clever subtractions to get a vector that is forced to be zero.

(f)

Problem. In the setting of the previous problem, show that $\mathrm{proj}_W(v)$ is the closest element of W to v, i.e. for any $w \in W$ we have $\|v - \mathrm{proj}_W(v)\| \leq \|v - w\|$.

Hint: Try writing $v - w$ as the sum of $v - \mathrm{proj}_W(v)$, which is in $W^\perp$, and $\mathrm{proj}_W(v) - w$ which is in W. Then expand out the norm squared of $v - w$.

Remark. In next week's problem set we will apply the above setup to $V = (C^0_{\mathrm{per}}(\mathbb{R},\mathbb{C}), \langle\cdot,\cdot\rangle_2)$, defining the Fourier coefficients and the Fourier series of $f \in C^0_{\mathrm{per}}(\mathbb{R},\mathbb{C})$. Indeed, for $f \in C^0_{\mathrm{per}}(\mathbb{R},\mathbb{C})$ and $k \in \mathbb{Z}$, define the n-the *Fourier coefficient* of f to be

$$\hat{f}(n) := \langle e_n, f\rangle_2 = \int_0^1 e^{-2\pi i n\theta} f(\theta) d\theta.$$

The *Fourier series* of $f = f(\theta)$ is the doubly infinite series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n\theta}.$$

In more detail, for $N, M \geq 0$ the (N, M) partial sum of the Fourier series is defined to be

$$\sum_{n=-M}^{N} \hat{f}(n) e_n,$$

which is the projection of f onto $\mathrm{span}\{e_{-M}, e_{-M+1}, \ldots, e_{N-1}, e_N\} \subset C^0_{\mathrm{per}}(\mathbb{R},\mathbb{C})$. Convergence of this series (pointwise, uniformly, or in L^2) is defined by considering the set of pairs (N, M) to be a directed set with $(N_1, M_1) \preceq (N_2, M_2)$ if

$N_1 \leq N_2$ and $M_1 \leq M_2$. In other words, for each $\varepsilon > 0$ there exists (N, M) such that for $n \geq N$ and $m \geq M$, the appropriate ε-condition is satisfied for $\sum_{k=-m}^{n} \hat{f}(k) e_k$.

Using the Stone–Weierstrass theorem, we will show next week that this series converges to f as a series of vectors in $(C^0_{\text{per}}(\mathbb{R}, \mathbb{C}), \langle \cdot, \cdot \rangle_2)$ (i.e. it " converges to f in the L^2 norm"). This result will also imply that the functions e_n form an orthonormal basis for $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$.