Problem 1. Let X be a compact metric space and let $A = \{a_n\}$ be a countable dense subset of X. For any $\delta > 0$, show that there exists M such that for any $x \in X$, we have $d(x, a_i) < \delta$ for some i with $i \leqslant i \leqslant M$.

Proof. Let $\delta > 0$ be given. By the denseness of A, for each $x \in X$ there exists some $a_i \in A$ such that $d(x, a_i) < \delta$. Therefore the countable union of δ-balls centered at each $a_i \in A$ covers X. Since X is compact, this open cover admits a finite subcover which consists of δ-balls centered at $a_{n_1}, a_{n_2}, \ldots, a_{n_k}$. Taking $\max\{n_1, \ldots, n_k\}$ gives our desired M and finishes the proof. □ □

Problem 2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \sin^2(n^n x)/\sqrt{n+2}$ for $n \geqslant 1$. Prove that the sequence $\{f_n\}$ is equicontinuous.

Proof. Let f_n be defined as above and let $\epsilon > 0$ be given. Notice that for all x, $0 \leqslant \sin^2(n^n x) \leqslant 1$, so $0 \leqslant f_n(x) \leqslant 1/\sqrt{n+2}$. If we pick N large enough such that $1/\sqrt{N+2} < \epsilon$ then for all $n \geqslant N$, $|f_n(x) - f_n(y)| < \epsilon$ is automatically satisfied. For the remaining $N-1$ terms (i.e., f_1 to f_{N-1}, since each is uniformly continuous (Lipschitz, in particular, since the derivative of $f_n(x)$ is $n^n \sin(n^n x)\cos(n^n x)/\sqrt{n+2}$, which is bounded), we can pick δ_n such that $|x - y| < \delta_i$ implies $|f_m(x) - f_m(y)| < \epsilon$. Define $\delta := \min\{\delta_1, \ldots, \delta_{N-1}\}$ and we indeed have $|x - y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon$ for all n. Hence the equicontinuity. □

Problem 3 (Pugh, 4.22). Suppose that $\mathcal{E} \subset C^0$ is equicontinuous and bounded.

(a) Prove that $\sup\{f(x) : f \in \mathcal{E}\}$ is a continuous function of x.

Proof. For convenience let f be the sup function. Let $\epsilon > 0$ be given. By the equicontinuity of $\mathcal{E}$, there exists $\delta > 0$ such that $|x - y| < \delta \implies |\tilde{f}(x) - \tilde{f}(y)| < \epsilon/2$ for all $\tilde{f} \in \mathcal{E}$. For any x, by the definition of supremum, there exists $\overline{f} \in \mathcal{E}$ such that $f(x) - \epsilon/2 < \overline{f}(x) \leqslant f(x)$. Therefore, for any y with $|x - y| < \delta$, we have

$$f(x) < \overline{f}(x) + \frac{\epsilon}{2} < \overline{f}(y) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \leqslant f(y) + \epsilon.$$

Interchanging x and y gives the other direction, namely $f(y) < f(x) + \epsilon$. Therefore f is continuous. □

(b) Show that (a) fails without equicontinuity.

Proof. Consider $\{f_n\}$ with $f_n(x) := 1 - x^n$. We have shown in class that this set of functions are not equicontinuous. The sup function is $f(x) = 1$ when $x \neq 1$ and $f(x) = 0$ when $x = 1$, clearly discontinuous. □

(c) Show that this continuous sup-property does not imply equicontinuity.

Proof. Still the same old example: let $\{f_n\}$ be defined as $f_n(x) := x^n$. Then the sup function is $f(x) = x$, continuous, but $\{f_n\}$ is not equicontinuous. □

(d) Assume that the continuous sup-property is true for each subset $\mathcal{F} \subset \mathcal{E}$. Is $\mathcal{E}$ equicontinuous?

Proof. No; simply consider the example given in (c). □