Problem 4a. Let $C^0_{\text{per}}(\mathbb{R},\mathbb{C})$ denote the subset of $C^0(\mathbb{R},\mathbb{C})$ consisting of periodic functions. Define a map $\Phi : C^0(S^1,\mathbb{C}) \to C^0_{\text{per}}(\mathbb{R},\mathbb{C})$ by

$$\Phi(f)(\theta) = f(e^{2\pi i\theta}).$$

Prove that Φ is well-defined and bijective.

Proof. If f is continuous then $\Phi(f)$ is too since it is the composition of two continuous functions. Furthermore, $\Phi(f)$ is periodic: for any $n \in \mathbb{Z}$, $\Phi(f(\theta+n)) = e^{2\pi i(\theta+n)} = e^{2\pi i\theta}e^{2\pi n} = e^{2\pi i\theta} = \Phi(f(\theta))$.

For injectivity, first let ρ be the branch cut along the positive real axis, starting from the origin. It follows that, for all $z \in S^1$ except $z = 1 = e^0$ we have $z = e^{\log(z)}$. Furthermore, $\log(z)$ must be purely imaginary: $e^{a+ib} = e^a e^{ib}$ and $|e^{a+ib}e^{ib}| = e^a|e^{ib}| = e^a$, and this $= 1$ if and only if $a = 0$. Therefore, if $f, g : S^1 \to \mathbb{C}$ agree at all points $e^{2\pi i\theta}$, they must agree everywhere.

For surjectivity, let $F \in C^0_{\text{per}}(\mathbb{R},\mathbb{C})$ be given. Define $f : S^1 \to \mathbb{C}$ by $f(u+iv) = F(\theta)$ for some $\theta \in \mathbb{R}$ and $u+iv = e^{2\pi i\theta}$. Since $e^{2\pi n} = 1$, $u+iv = e^{2\pi i\theta} = e^{2\pi i(\theta+n)}$ so f is well-defined. ☐

Problem 4b. Prove that Φ is linear and preserves the uniform norm.

Proof. Linearity follows from the fact that

$$\begin{aligned}\Phi(f_1+\lambda f_2)(\theta) &= (f_1+\lambda f_2)(2^{2\pi i\theta}) \\ &= f_1(e^{2\pi i\theta}) + \lambda f_2(e^{2\pi i\theta}) \\ &= \Phi(f_1(\theta)) + \lambda\Phi(f_2(\theta)).\end{aligned}$$

For preservation of uniform norm, let $f, g \in C^0(S^1,\mathbb{C})$. Suppose for some $z \in S^1$ we have $|f(z)-g(z)| = x$. If $z = 1$ or $z = -1$ then setting $\theta = 1$ and $\theta = 0.5$ would do the job, as $e^{2\pi i} = 1$ and $e^{\pi i} = -1$. Otherwise, by (a), there exists some purely imaginary number $\log(z)$ such that $e^{\log(z)} = z$. Setting $\theta := \log(z)/2\pi i$ gives our desired equality, namely $|f(z)-g(z)| = |f(e^{2\pi i\theta}) - g(e^{2\pi i\theta})|$. Therefore $\{|f(z)-g(z)| : z \in S^1\} \subset \{|f(e^{2\pi i\theta}) - g(e^{2\pi i\theta})| : \theta in \mathbb{R}\}$. For the other direction ($\supset$), simply notice that $|e^{2\pi i\theta}| = 1$ for all $\theta \in \mathbb{R}$. Thus the two sets are equal. ☐

Problem 4c. Show that $\langle\cdot,\cdot\rangle_2$ defined by

$$\langle f, g\rangle_2 := \int_0^1 \overline{f(\tilde{x})}g(\tilde{x})\,\mathrm{d}\tilde{x}$$

is a complex inner product on $C^0_{\text{per}}(\mathbb{R},\mathbb{C})$, i.e., show that the following axioms are satisfied for $\langle\cdot,\cdot\rangle_2$:

(1) $\langle f, g\rangle = \overline{\langle g, f\rangle}$ for all f, g,

(2) $\langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle$ for all f_1, f_2, g,

(3) $\langle f, cg\rangle = c\langle f, g\rangle$ for all c, f, g, and

(4) $\langle f, f\rangle$ is non-degenerate.

Proof. (1) follows from the fact that the conjugate of $\overline{f(x)}g(x)$ is $f(x)\overline{g(x)}$. (2) is because $(\overline{(f_1+f_2)(x)})g(x)$ is the same as $\overline{f_1(x)}g(x) + \overline{f_2(x)}g(x)$. (3) is trivial. The "$\geqslant 0$" part of (4) is obvious as $\overline{f(x)}f(x)$ gives us the modulus of a complex number which is never negative. Now suppose $\langle f, f\rangle = 0$. Since f is continuous, so is its conjugate, and thus their product $\overline{f}f$. If the integral of a nonnegative, continuous function evaluates to 0, then it must be the case that $\overline{f}f \equiv 0$ (otherwise there exists some interval (a,b) in which $f(x) \geqslant m$ for some m and all $x \in (a,b)$ by the property of continuity). Thus $f \equiv 0$. ☐

Problem 4d. Show that $\{e^{2\pi in\theta:n\in\mathbb{Z}}\}$ is orthornormal.

Proof. If $m \neq n$, then

$$\left\langle e^{2\pi n\theta}, e^{2\pi m\theta}\right\rangle = \int_0^1 e^{-2\pi n\theta}e^{2\pi m\theta}\, \mathrm{d}\theta = \int_0^1 e^{2\pi(m-n)\theta}\, \mathrm{d}\theta = \int_0^1 0\, \mathrm{d}\theta = 0,$$

and if $m = n$, then $\left\langle e^{2\pi m\theta}, e^{2\pi n\theta}\right\rangle = \int_0^1 e^0\, \mathrm{d}\theta = 1.$ ☐

Problem 4e. Show that $\mathrm{proj}_W(v) \in W$ and $v - \mathrm{proj}_W(v) \in W^\perp$. Further show that this characterizes $\mathrm{proj}_W(v)$, i.e., the orthogonal projection of v is unique.

Proof. The first part is trivial since the projection of v onto W is simply a linear combination of $w_1, \ldots, w_n$. To show $v - \mathrm{proj}_W(v) \in W^\perp$, it suffices to check that it is orthogonal to any basis of W. Pick any $w_i \in W$. Then,

$$\begin{aligned}\langle w_i, v - \mathrm{proj}_W(v)\rangle &= \langle w_i, v\rangle - \langle w_i, \mathrm{proj}_W(v)\rangle \\ &= \langle w_i, v\rangle - \langle w_1, v\rangle\langle w_i, w_1\rangle - \cdots - \langle w_n, v\rangle\langle w_i, w_n\rangle \\ &= \langle w_i, v\rangle - \langle w_i, w_i\rangle = 0.\end{aligned}$$

For uniqueness, suppose $v = v_1 + v_2 = v_1' + v_2'$ with $v_1, v_1' \in W$ and $v_2, 2' \in W^\perp$. Then $v_1 - v_1' = v_2' - v_2$, so

$$\|v_1 - v_1'\|^2 = \langle v_1 - v_1', v_1 - v_1'\rangle = \langle v_1 - v_1', v_2' - v_2\rangle = 0$$

as the first component is in W and the second in $W^\perp$. Hence $v_1 = v_1'$ and likewise $v_2 = v_2'$. Uniqueness follows. ☐

Problem 3f. Now show that $\mathrm{proj}_W(v)$ is the closest element of W to v.

Proof. Let $w \in W$ be given. We can rewrite $v - w$ as $(v - \mathrm{proj}_W(v)) + (\mathrm{proj}_W(v) - w)$, where the first parenthesis represents an element of $W^\perp$ and the second of W. Then,

$$\begin{aligned}\|v - w\|^2 &= \langle(\text{first}) + (\text{second}), (\text{first}) + (\text{second})\rangle \\ &= \|v - \mathrm{proj}_W(v)\|^2 + \|\mathrm{proj}_W(v) - w\|^2 + 0 + \overline{0}.\end{aligned}$$

From this the original claim becomes clear, and we also see the = can only be obtained when $w = \mathrm{proj}_W(v)$. ☐