MATH 425b Homework 7

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Problem 1. Let X be a compact metric space and let $A = \{a_n\}$ be a countable dense subset of X. For any $\delta > 0$, show that there exists M such that for any $x \in X$, we have $d(x, a_i) < \delta$ for some i with $i \leqslant i \leqslant M$.

Proof. Let $\delta > 0$ be given. By the denseness of A, for each $x \in X$ there exists some $a_i \in A$ such that $d(x, a_i) < \delta$. Therefore the countable union of δ-balls centered at each $a_i \in A$ covers X. Since X is compact, this open cover admits a finite subcover which consists of δ-balls centered at $a_{n_1}, a_{n_2}, \ldots, a_{n_k}$. Taking $\max\{n_1, \ldots, n_k\}$ gives our desired M and finishes the proof. □

Problem 2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \sin^2(n^n x)/\sqrt{n+2}$ for $n \geqslant 1$. Prove that the sequence $\{f_n\}$ is equicontinuous.

Proof. Let f_n be defined as above and let $\epsilon > 0$ be given. Notice that for all x, $0 \leqslant \sin^2(n^n x) \leqslant 1$, so $0 \leqslant f_n(x) \leqslant 1/\sqrt{n+2}$. If we pick N large enough such that $1/\sqrt{N+2} < \epsilon$ then for all $n \geqslant N$, $|f_n(x) - f_n(y)| < \epsilon$ is automatically satisfied. For the remaining $N-1$ terms i.e., f_1 to f_{N-1}, since each is uniformly continuous (Lipschitz, in particular, since the derivative of $f_n(x)$ is $n^n \sin(n^n x)\cos(n^n x)/\sqrt{n+2}$, which is finite), we can pick δ_n such that $|x-y| < \delta_i$ implies $|f_n(x) - f_n(y)| < \epsilon$. Define $\delta := \min\{\delta_1, \ldots, \delta_{N-1}\}$ and we indeed have $|x - y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon$ for all n. Hence the equicontinuity. □

Problem 3 (Pugh, 4.22). Give an example of a sequence of smooth equicontinuous functions $f_n : [a, b] \to \mathbb{R}$ whose derivatives are not uniformly bounded.

Solution

Simply consider $\{f_n\}$ as mentioned in the previous problem. They are smooth because $\sin(x)$ is. However, the derivatives are not uniformly bounded (even though each one is finite):

$$f_n'(x) = \frac{2n^n \sin(n^n x)\cos(n^n x)}{\sqrt{n+2}}$$

which $\to \infty$ as $n \to \infty$ because n^n outgrows everything else.

Problem 4a. Let $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ denote the subset of $C^0(\mathbb{R}, \mathbb{C})$ consisting of periodic functions. Define a map $\Phi : C^0(S^1, \mathbb{C}) \to C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ by

$$(\Phi(f))(\theta) = f(e^{2\pi i \theta}).$$

Prove that Φ is well-defined and bijective.

Proof. $\Phi(f)$ is clearly continuous as it is the composition of a continuous exponential function with a continuous $f \in C^0(S_1, \mathbb{C})$. It is periodic because $(\Phi(f))(\theta + n) = f(e^{2\pi i(\theta+n)}) = f(e^{2\pi i\theta} \cdot e^{2\pi in}) = f(e^{2\pi i\theta})$.

For injectivity, notice that any $z \in S_1$ can be written as e^w, where $w = \log(z)$ with any logarithm function defined with a branch cut ρ not containing z. Write $w = a + bi$. Then $e^w = e^a e^{ib}$ and its modulus $|e^w| = |e^a||e^{ib}| = |e^a|$. Since $z \in S^1$ we must have $|e^a| = 1$, of which the only real solution is $a = 0$. Therefore f, g agree on all points of form $e^{2\pi i\theta}$, $\theta \in \mathbb{R}$, i.e., $\Phi(f) = \Phi(g)$, they must agree, in particular, on all points in S^1, since all points in S^1 are of this form. Thus Φ is injective.

For surjectivity, let $F \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ be given. Pick any $\theta \in \mathbb{R}$. We define a function $f : S^1 \to \mathbb{R}$ by $f(u+iv) = F(\theta)$ where $e^{2\pi i\theta} = u + iv$. It follows that f is well-defined for the periodicity of F, i.e., $F(\theta) = F(\theta + n), n \in \mathbb{Z}$ not violated since $e^{2\pi i\theta} = e^{2\pi i(\theta+n)}$. It remains to show to show that f is continuous. Indeed, given $u + iv \in S^1$, consider the ray pointing towards the opposite direction, i.e., from the origin to $-u - iv$ (or any direction not containing $u + iv$). Then

$$f(u + iv) = F(\theta) = F((\log(u + iv))/2\pi i)$$

can be written as a composition of continuous functions $(f(z) = F \circ (\log(z)/2\pi i))$ and is therefore continuous on some open subset of $(\mathbb{C} \setminus \rho) \cap S^1 \subset S^1$. Thus, $f \in C^0(S^1, \mathbb{C})$ and by construction $\Phi(f) = F$, so Φ is surjective. □

Problem 4b. Prove that Φ is linear and preserves the uniform norm.

Proof. Linearity follows from the fact that

$$\begin{aligned}\Phi(f_1 + \lambda f_2)(\theta) &= (f_1 + \lambda f_2)(2^{2\pi i\theta}) \\ &= f_1(e^{2\pi i\theta}) + \lambda f_2(e^{2\pi i\theta}) \\ &= \Phi(f_1(\theta)) + \lambda\Phi(f_2(\theta)).\end{aligned}$$

For preservation of uniform norm, let $f, g \in C^0(S^1, \mathbb{C})$. Suppose for some $z \in S^1$ we have $|f(z) - g(z)| = x$. If $z = 1$ or $z = -1$ then setting $\theta = 1$ and $\theta = 0.5$ would do the job, as $e^{2\pi i} = 1$ and $e^{\pi i} = -1$. Otherwise, by (a), there exists some purely imaginary number $\log(z)$ such that $e^{\log(z)} = z$. Setting $\theta := \log(z)/2\pi i$ gives our desired equality, namely $|f(z) - g(z)| = |f(e^{2\pi i\theta}) - g(e^{2\pi i\theta})|$. Therefore $\{|f(z) - g(z)| : z \in S^1\} \subset \{|f(e^{2\pi i\theta}) - g(e^{2\pi i\theta})| : \theta \in \mathbb{R}\}$. For the other direction ($\supset$), simply notice that $|e^{2\pi i\theta}| = 1$ for all $\theta \in \mathbb{R}$. Thus the two sets are equal. □

Problem 4c. Show that $\langle\cdot,\cdot\rangle_2$ defined by

$$\langle f, g\rangle_2 := \int_0^1 \overline{f(\tilde{x})} g(\tilde{x})\, \mathrm{d}\tilde{x}$$

is a complex inner product on $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$, i.e., show that the following axioms are satisfied for $\langle\cdot,\cdot\rangle_2$:

(1) $\langle f, g\rangle = \overline{\langle g, f\rangle}$ for all f, g,

(2) $\langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle$ for all f_1, f_2, g,

(3) $\langle f, cg\rangle = c\langle f, g\rangle$ for all c, f, g, and

(4) $\langle f, f\rangle$ is non-degenerate.

Proof. (1) follows from the fact that the conjugate of $\overline{f(x)}g(x)$ is $f(x)\overline{g(x)}$. (2) is because $(\overline{(f_1+f_2)(x)})g(x)$ is the same as $\overline{f_1(x)}g(x)+\overline{f_2(x)}g(x)$. (3) is trivial. The "$\geqslant 0$" part of (4) is obvious as $\overline{f(x)}f(x)$ gives us the modulus of a complex number which is never negative. Now suppose $\langle f,f\rangle = 0$. Since f is continuous, so are its conjugate and their product $\overline{f}f$. If the integral of a nonnegative, continuous function evaluates to 0, then it must be the case that $\overline{f}f \equiv 0$ (otherwise there exists some interval (a,b) in which $f(x)\geqslant m$ for some m and all $x\in(a,b)$ by the property of continuity). Thus $f\equiv 0$. □

Problem 4d. Show that $\{e^{2\pi in\theta:n\in\mathbb{Z}}\}$ is orthornormal.

Proof. If $m\neq n$, then

$$\left\langle e^{2\pi n\theta}, e^{2\pi m\theta}\right\rangle = \int_0^1 e^{-2\pi n\theta}e^{2\pi m\theta}\,\mathrm{d}\theta = \int_0^1 e^{2\pi(m-n)\theta}\,\mathrm{d}\theta = \int_0^1 0\,\mathrm{d}\theta = 0,$$

and if $m=n$, then $\left\langle e^{2\pi m\theta}, e^{2\pi n\theta}\right\rangle = \int_0^1 e^0\,\mathrm{d}\theta = 1$. □

Problem 4e. Show that $\mathrm{proj}_W(v)\in W$ and $v-\mathrm{proj}_W(v)\in W^\perp$. Further show that this characterizes $\mathrm{proj}_W(v)$, i.e., the orthogonal projection of v is unique.

Proof. The first part is trivial since the projection of v onto W is simply a linear combination of $w_1,\ldots,w_n$. To show $v-\mathrm{proj}_W(v)\in W^\perp$, it suffices to check that it is orthogonal to any basis of W. Pick any $w_i\in W$. Then,

$$\begin{aligned}\langle w_i, v-\mathrm{proj}_W(v)\rangle &= \langle w_i,v\rangle - \langle w_i,\mathrm{proj}_W(v)\rangle\\ &= \langle w_i,v\rangle - \langle w_1,v\rangle\langle w_i,w_1\rangle - \cdots - \langle w_n,v\rangle\langle w_i,w_n\rangle\\ &= \langle w_i,v\rangle - \langle w_i,w_i\rangle = 0.\end{aligned}$$

For uniqueness, suppose $v=v_1+v_2=v_1'+v_2'$ with $v_1,v_1'\in W$ and $v_2,2'\in W^\perp$. Then $v_1-v_1'=v_2'-v_2$, so

$$\|v_1-v_1'\|^2 = \langle v_1-v_1', v_1-v_1'\rangle = \langle v_1-v_1', v_2'-v_2\rangle = 0$$

as the first component is in W and the second in $W^\perp$. Hence $v_1=v_1'$ and likewise $v_2=v_2'$. Uniqueness follows. □

Problem 3f. Now show that $\mathrm{proj}_W(v)$ is the closest element of W to v.

Proof. Let $w\in W$ be given. We can rewrite $v-w$ as $(v-\mathrm{proj}_W(v))+(\mathrm{proj}_W(v)-w)$, where the first parenthesis represents an element of $W^\perp$ and the second of W. Then,

$$\begin{aligned}\|v-w\|^2 &= \langle(\text{first})+(\text{second}),(\text{first})+(\text{second})\rangle\\ &= \|v-\mathrm{proj}_W(v)\|^2 + \|\mathrm{proj}_W(v)-w\|^2 + 0 + \overline{0}.\end{aligned}$$

From this the original claim becomes clear, and we also see the = can only be obtained when $w=\mathrm{proj}_W(v)$. □