HOMEWORK, WEEK 8

This assignment is due Monday, March 15 in lecture. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

In this problem set we will continue our exploration of Fourier series.

(1) Recall the following definitions from the end of the previous problem set: for $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ and $k \in \mathbb{Z}$, define the n-the *Fourier coefficient* of f to be

$$\hat{f}(n) := \langle e_n, f \rangle_2 = \int_0^1 e^{-2\pi i n\theta} f(\theta) d\theta.$$

The *Fourier series* of $f = f(\theta)$ is the doubly infinite series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n\theta}.$$

In more detail, for $N, M \geq 0$ the (N, M) partial sum of the Fourier series is defined to be

$$\sum_{n=-M}^{N} \hat{f}(n) e_n,$$

which is the projection of f onto $\text{span}\{e_{-M}, e_{-M+1}, \dots, e_{N-1}, e_N\} \subset C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$. Convergence of this series (pointwise, uniformly, or in L^2) is defined by considering the set of pairs (N, M) to be a directed set with $(N_1, M_1) \preceq (N_2, M_2)$ if $N_1 \leq N_2$ and $M_1 \leq M_2$. In other words, for each $\varepsilon > 0$ there exists (N, M) such that for $n \geq N$ and $m \geq M$, the appropriate ε-condition is satisfied for $\sum_{k=-m}^{n} \hat{f}(k) e_k$.

Problem. Prove that for $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$, the Fourier series of f converges to f in the L^2 norm.

Hint: Given ε, using the Stone–Weierstrass theorem, find an element p of $\mathcal{A} := \text{span}\{\cdots, e_{-2}, e_{-1}, e_0, e_1, e_2, \cdots\}$ that is close enough to f in the norm $\|\cdot\|_{\text{sup}}$ (make sure you check the conditions of the theorem, and figure out "close enough" at the end of the proof). Since p is a finite linear combination of the functions e_n, there exist N, M such that p is in $\text{span}\{e_{-M}, e_{-M+1}, \cdots, e_{N-1}, e_N\}$, which itself is contained in

$$\text{span}\{e_{-m}, e_{-m+1}, \cdots, e_{n-1}, e_n\}$$

for any $n \geq N$, $m \geq M$. For such n and m, try to bound the L^2 norm of $f - \sum_{k=-m}^{n} \hat{f}(k) e_k$, and fix "close enough" earlier in the proof accordingly.

It follows that the exponential functions $\{e_n\}_{n \in \mathbb{Z}}$ form an orthonormal basis (in the analytic sense) for $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ with the L^2 inner product.

Remark. The metric space $(C^0_{\text{per}}(\mathbb{R},\mathbb{C}), d_{L^2})$ is not complete, so it is not a Hilbert space (if you want, try to find a sequence of continuous functions converging in L^2 to a discontinuous function). A sign of this incompleteness is that not all square-summable sequences of complex numbers arise as the Fourier coefficients of a continuous function.

The more natural setting for L^2 convergence of Fourier series is the Hilbert space $L^2(S^1,\mathbb{C})$ of functions $f : S^1 \to \mathbb{C}$ with $|f|^2$ Lebesgue integrable, modulo almost-everywhere equality. This Hilbert space contains $C^0_{\text{per}}(\mathbb{R},\mathbb{C})$ as a (dense) subspace. The standard proofs of L^2 convergence work with this space, and thus do not use continuity or the Stone–Weierstrass theorem (which is about continuous functions). Like any (infinite-dimensional separable) Hilbert space over $\mathbb{C}$, $L^2(S^1,\mathbb{C})$ is isomorphic to the space ℓ^2 of square-summable sequences of complex numbers (this result is an analogue of the fact "all n-dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$ are isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$" when $n = \infty$).

(2) Let $f \in C^0_{\text{per}}(\mathbb{R},\mathbb{C})$, which we can identify with $C^0(S^1,\mathbb{C})$ as in last week's problem set. For $n \in \mathbb{Z}$, recall that

$$\hat{f}(n) = \langle e_n, f\rangle_{L^2} = \int_0^1 f(x)e^{-2\pi i n x}dx$$

is the n^{th} Fourier coefficient of f.

Problem. Prove that the doubly infinite series of real numbers $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ converges, with limit equal to $\|f\|^2_{L^2}$.

Hint: For given integers $M, N \geq 0$, let $S_{M,N}(f)$ denote the partial sum of the Fourier series of f from $-M$ to N, an element of $\mathbb{C}^0_{\text{per}}(\mathbb{R},\mathbb{C})$. Show that $S_{M,N}(f)$ and $f - S_{M,N}(f)$ are orthogonal (i.e. their inner product is zero); conceptually, it might help to write $S_{M,N}(f)$ as an orthogonal projection of f onto a finite-dimensional subspace of $C^0_{\text{per}}(\mathbb{R},\mathbb{C})$ as in last week's homework. Deduce that

$$\|f\|^2_{L^2} = \|f - S_{M,N}(f)\|^2_{L^2} + \|S_{M,N}(f)\|^2_{L^2}.$$

Show that $\|S_{M,N}(f)\|^2_{L^2}$ is a partial sum of the real-number series $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$, which you want to show converges to $\|f\|^2_{L^2}$. Now, given ε, use L^2 convergence to show that there exist M, N such that for $m \geq M$ and $n \geq N$, the quantity $\|S_{M,N}(f)\|^2_{L^2}$ is closer than ε to $\|f\|^2_{L^2}$.

Remark. This result is known as Parseval's identity, and it holds more generally when f is any element of $L^2(S^1,\mathbb{C})$ (i.e. when f is Lebesgue square-integrable on S^1). Let $\ell^2_{\mathbb{C}}$ denote the $\mathbb{C}$-vector space of doubly-infinite sequences $\{a_n\}_{n\in\mathbb{Z}}$ where $a_n \in \mathbb{C}$ and $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$ (such sequences are called "square-summable"). There is an inner product $\langle\cdot,\cdot\rangle_{\ell^2}$ on $\ell^2_{\mathbb{C}}$ defined by $\langle(a_n),(b_n)\rangle_{\ell^2} := \sum_{n=-\infty}^{\infty} \bar{a_n} b_n$ (the inequality $|a_n b_n| \leq (1/2)(|a_n|^2 + |b_n|^2)$ ensures this is well-defined).

By the above problem, if $f \in C^0(S^1,\mathbb{C})$, then $\{\hat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{C}}$, so we have a linear transformation

$$\mathcal{F} : C^0(S^1,\mathbb{C}) \to \ell^2_{\mathbb{C}}$$

sending f to $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$. The transformation $\mathcal{F}$ is injective but not surjective (as mentioned on the previous problem set). The corresponding linear transformation

$$\mathcal{F} : L^2(S^1, \mathbb{C}) \to \ell^2_{\mathbb{C}}$$

is an isomorphism. Parseval's identity says that $\mathcal{F}$ preserves L^2 norms, in the sense that $\|\mathcal{F}(f)\|_{\ell^2} = \|f\|_{L^2}$ (this holds for all $f \in L^2(S^1, \mathbb{C})$). By the polarization identity for complex inner product spaces, it follows that $\mathcal{F}$ preserves L^2 inner products, in the sense that $\langle \mathcal{F}(f), \mathcal{F}(g)\rangle_{\ell^2} = \langle f, g\rangle_{L^2}$ for all $f, g \in L^2(S^1, \mathbb{C})$.

Abstractly, you can think of $\ell^2_{\mathbb{C}}$ as $L^2(\mathbb{Z}, \mathbb{C})$ where $\mathbb{Z}$ is equipped with the "counting measure" (measure of a set := its cardinality). In fact, both S^1 and $\mathbb{Z}$ are (locally compact) topological abelian groups, and there exists a duality $G \leftrightarrow \hat{G}$ ("Pontryagin duality") for such groups. In general, there is a Fourier transform relating functions on G with functions on $\hat{G}$. In the case we've been considering, we have $\hat{S^1} = \mathbb{Z}$ (and $\hat{\mathbb{Z}} = S^1$), so functions on $\mathbb{Z}$ should correspond to functions on S^1, and this is true at least for L^2 functions. Fourier transforms for non-periodic functions arise from the Pontryagin duality relationship $\hat{\mathbb{R}} = \mathbb{R}$.

(3)

Problem. Show that if $f \in C^0(S^1, \mathbb{C})$ (or equivalently $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$), then

$$\lim_{n \to \pm\infty} \hat{f}(n) = 0.$$

Hint: The idea is that this result should follow from the fact that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ converges, so its general term approaches zero. Since the series are doubly infinite, though, it's best to give a brief argument that the general term of a doubly infinite series tends to zero as $n \to \infty$ and as $n \to -\infty$ (you can probably just say the $n \to -\infty$ case is analogous to the $n \to \infty$ case to avoid duplication but you should do the $n \to \infty$ case).

Remark. This result holds with the same proof for $f \in L^2(S^1, \mathbb{C})$. Less trivially, it also holds for $f \in L^1(S^1, \mathbb{C})$, where it is known as the Riemann–Lebesgue lemma for Fourier series (the more standard Riemann–Lebesgue lemma is for Fourier transforms). Since this result is what most people think when you say "Riemann–Lebesgue lemma," I prefer saying "Lebesgue's criterion for Riemann integrability" when referring to Theorem 23 in Section 3.2 of Pugh.

(4) Define $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ by setting $f(x) = (1/2 - x)^2$ for $x \in [0, 1]$ and extending f periodically to all of $\mathbb{R}$ (since f is continuous on $[0, 1]$ and $f(0) = f(1) = 1/4$, the periodic extension of f is continuous on all of $\mathbb{R}$).

Problem. Compute the Fourier coefficients $\hat{f}(n)$ for all $n \in \mathbb{Z}$, and write down the corresponding Fourier series as a doubly infinite sum of exponentials. Rewrite the series as a singly infinite sum of cosines.

Hint: this is an integration exercise; it's important to be able to work with Fourier series concretely. Treat the cases $n = 0$ and $n \neq 0$ separately, using integration by parts twice when $n \neq 0$. To rewrite the series in terms of cosines, try combining terms n and $-n$ for $n \neq 0$ and using identities relating cosines and exponentials.

Remark. This Fourier series converges to f in L^2 by what we've proved; you'll show uniform convergence in the next problem. It might be informative to draw the graph of f as a periodic function; it looks a bit like the usual sketch one might draw for water waves. If $f(x)$ represents a sound wave ($f(x)$ = air pressure at some given point at time x), then the coefficients of the cosine functions represent the amplitudes of the "partials" present in the sound. This phenomenon should be familiar to musicians.

(5) Now we consider absolute convergence of Fourier series, which is a relatively strong condition.

Problem. Let $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ and suppose that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ converges (this is the series of absolute values for $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$). Prove that the Fourier series of f converges uniformly to f. Deduce that the Fourier series of $f(x) = (1/2 - x)^2$ converges uniformly to f.

Hint: First use the Weierstrass M-test to show that the Fourier series converges absolutely uniformly (and thus uniformly to some function g). It's good to mention that the M-test extends without issue to the case of doubly infinite series (you don't have to prove this here, though). Then use L^2 convergence of Fourier series and uniqueness of limits in metric spaces; make sure to give the appropriate details.