Problem 0.0.1

Prove that for $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$, the Fourier series of f converges to f in the L^2 norm.

Proof. Define $\mathcal{A} := \operatorname{span}\{e^{2\pi inx}, n \in \mathbb{Z}\}$. Since $\mathcal{A}$ is a $\mathbb{C}$-function algebra (with conjugation $\overline{\sum e^{2\pi inx}} = \sum e^{-2\pi inx}$), vanishes nowhere (exponential functions are never 0), and separates points ($e^{2\pi ix}$ has period 1) between $[0,1]$, by the Stone-Weierstraß Theorem it is dense in $(C^0_{\text{per}}(\mathbb{R}, \mathbb{C}), \|\cdot\|_{\sup})$.

Let $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ be given. We can then find $g \in \mathcal{A}$, a linear combination of e_n's, such that $\|f - g\|_{\sup} < \epsilon$. Note that this linear combination is finite; the indices of the e_n's are bounded by some $-M$ and N where $M, N \in \mathbb{N}$. Since g may or may not be the orthogonal projection of f onto $\operatorname{span}\{e_{-M}, \dots, e_N\}$ (so that $\|f - g\|_\infty$ may or may not be minimized), we have

$$\left\| f - \sum_{k=-M}^{N} \hat{f}(k) e_k \right\| \leqslant \|f - g\|_{\sup} < \sqrt{\epsilon}.$$

The same holds for any $m \geqslant M$ and $n \geqslant N$, as g may or may not be the orthogonal projection of f onto $\operatorname{span}\{e_{-m}, \dots, e_{-M}, \dots, e_N, \dots, e_n\}$. Therefore, for all (m,n) with $(M,N) \leq (m,n)$, we have

$$\begin{aligned}
\left\| f - \sum_{k=-m}^{n} \hat{f}(k) e_k \right\|_{L^2} &= \left(\int_0^1 \left| f(\tilde{x}) - \sum_{k=-m}^{n} \hat{f}(k) e_k(\tilde{x}) \right|^2 \mathrm{d}\tilde{x} \right)^{1/2} \\
&\leqslant \left(\int_0^1 |f(\tilde{x}) - g(\tilde{x})|^2 \, \mathrm{d}\tilde{x} \right)^{1/2} \\
&< \left(\int_0^1 \epsilon^2 \, \mathrm{d}\tilde{x} \right)^{1/2} = \epsilon,
\end{aligned}$$

and thus the Fourier series of f converges to f in the L^2 norm. □

Problem 0.0.2

Let $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$, which we can identify with $C^0(S^1, \mathbb{C})$. For $n \in \mathbb{Z}$, recall that

$$\hat{f}(n) = \langle e_n, f \rangle_{L^2} := \int_0^1 f(x) e^{-2\pi in\tilde{x}} \, \mathrm{d}\tilde{x}$$

is the n^{th} Fourier coefficient of f. Prove that the doubly infinite series of real numbers $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ converges, with limit equal to $\|f\|^2_{L^2}$.

Proof. Recall that $\{e_n\}$ is orthonormal and so are its subsets. If we define $S_n(f) := \sum_{k=-n}^{n} \hat{f}(k) e_k$, then it becomes clear that $S_n(f)$ is the orthogonal projection of f onto $\operatorname{span}\{e_{-n}, \dots, e_n\}$. Therefore by the previous problem set we immediately know $f - S_n(f)$ is orthogonal to $S_n(f)$. Notice that

$$\begin{aligned}
\|f\|^2_{L^2} = \langle f, f \rangle_{L^2} &= \langle f - S_n(f) + S_n(f), f - S_n(f) + S_n(f) \rangle_{L^2} \\
&= \langle f - S_n(f), f - S_n(f) \rangle_{L^2} + \langle S_n(f), S_n(f) \rangle_{L^2} + \underbrace{2\mathfrak{Re} \langle f - S_n(f), f \rangle}_{=0} \\
&= \|f - S_n(f)\|^2_{L^2} + \|S_n(f)\|^2_{L^2}.
\end{aligned}$$

From problem 1, we know that, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ sufficiently large such that $\|f - S_n(f)\|^2_{L^2} < \epsilon$ whenever $n \geqslant N$. When this happens, the equation above suggests $\|f\|^2_{L^2} - \epsilon < \|S_n(f)\|^2_{L^2} < \|f\|^2_{L^2}$. Therefore as

$n \to \infty$ we have $\|S_n(f)\|_{L^2}^2 \to \|f\|_{L^2}^2$. On the other hand, by generalized Pythagorean Theorem,

$$\begin{aligned}\|S_n(f)\|_{L^2}^2 = \left\| \sum_{k=-n}^{n} \hat{f}(k) e_k \right\| &= \left\langle \sum_{k=-n}^{n} \hat{f}(k) e_k, \sum_{k=-n}^{n} \hat{f}(k) e_k \right\rangle_{L^2} \\ &= \sum_{j=-n}^{n} \sum_{m=-n}^{n} \hat{f}(j) \hat{f}(m) \underbrace{\langle e_j, e_m \rangle_{L^2}}_{=\delta(j,m)} \\ &= \sum_{j=-n}^{n} \hat{f}(j)^2 = \sum_{j=-n}^{n} |\hat{f}(j)|^2,\end{aligned}$$

and letting $n \to \infty$ gives $\|S_n(f)\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$. Therefore the two must equal, i.e.,

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \|f\|_{L^2}^2. \qquad \text{(Parseval's Identity)}$$ □

Problem 0.0.3

Show that if $f \in C^0(S^1, \mathbb{C})$ then

$$\lim_{n \to \pm\infty} \hat{f}(n) = 0.$$

Proof. Since each $|\hat{f}(n)|^2 \geqslant 0$ and $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ converges, do do $\sum_{n=0}^{-\infty} |\hat{f}(n)|^2$ and $\sum_{n=1}^{\infty} |\hat{f}(n)|^2$. Therefore

$$\lim_{n \to \pm\infty} |\hat{f}(n)|^2 = 0 \implies \lim_{n \to \pm\infty} |\hat{f}(n)| = 0.$$ □

Problem 0.0.4

Define $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ by setting $f(x) = (1/2 - x)^2$ for $x \in [0,1]$ and extending f periodically to all or $\mathbb{R}$. Compute the Fourier coefficients $\hat{f}(n)$ for all $n \in \mathbb{Z}$ and write down the corresponding Fourier series as a doubly infinite sum of exponentials. Rewrite the series as a single infinite sum of cosines.

Solution

If $n = 0$, $\hat{f}(n)$ is simply

$$\int_0^1 e^0 (1/2 - \theta)^2 \, \mathrm{d}\theta = -\frac{1}{3}(1/2 - \theta)^3 \Big|_{\theta=0}^{1} = \frac{1}{12}.$$

Before computing $\hat{f}(n)$ for nonzero n's, we first compute the indefinite integral $\int e^{-2\pi i n \theta} f(\theta) \, \mathrm{d}\theta$.

$$\int e^{-2\pi i n \theta} (1/2 - \theta)^2 \, \mathrm{d}\theta \qquad \begin{bmatrix} u = (1/2 - \theta)^2 & \mathrm{d}u = 2\theta - 1 \\ \mathrm{d}v = e^{-2\pi i n \theta} \mathrm{d}\theta & \begin{aligned} v &= e^{-2\pi i n \theta}/(-2\pi i n) \\ &= i e^{-2\pi i n \theta}/(2\pi n) \end{aligned} \end{bmatrix}$$

$$= \frac{i e^{-2\pi i n \theta}}{2\pi n}(1/2 - \theta)^2 - \int \frac{i e^{-2\pi i n \theta}}{2\pi n}(2\theta - 1) \, \mathrm{d}\theta.$$