MATH 425b Homework 8

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Problem 1

Prove that for $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$, the Fourier series of f converges to f in the L^2 norm.

Proof. Define $\mathcal{A} := \text{span}\{e^{2\pi i n x}, n \in \mathbb{Z}\}$. Since $\mathcal{A}$ is a $\mathbb{C}$-function algebra (with conjugation $\overline{\sum e^{2\pi i n x}} = \sum e^{-2\pi i n x}$), vanishes nowhere (exponential functions are never 0), and separates points ($e^{2\pi i x}$ has period 1) between $[0,1]$, by the Stone-Weierstraß Theorem it is dense in $(C^0_{\text{per}}(\mathbb{R}, \mathbb{C}), \|\cdot\|_{\text{sup}})$.
Let $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ be given. We can then find $g \in \mathcal{A}$, a linear combination of e_n's, such that $\|f - g\|_{\text{sup}} < \epsilon$. Note that this linear combination is finite; the indices of the e_n's are bounded by some $-M$ and N where $M, N \in \mathbb{N}$. Since g may or may not be the orthogonal projection of f onto $\text{span}\{e_{-M}, \ldots, e_N\}$ (so that $\|f - g\|_\infty$ may or may not be minimized), we have

$$\left\| f - \sum_{k=-M}^{N} \hat{f}(k) e_k \right\| \leqslant \|f - g\|_{\text{sup}} < \sqrt{\epsilon}.$$

The same holds for any $m \geqslant M$ and $n \geqslant N$, as g may or may not be the orthogonal projection of f onto $\text{span}\{e_{-m}, \ldots, e_{-M}, \ldots, e_N, \ldots, e_n\}$. Therefore, for all (m,n) with $(M,N) \preceq (m,n)$, we have

$$\begin{aligned}
\left\| f - \sum_{k=-m}^{n} \hat{f}(k) e_k \right\|_{L^2} &= \left(\int_0^1 \left| f(\tilde{x}) - \sum_{k=-m}^{n} \hat{f}(k) e_k(\tilde{x}) \right|^2 \mathrm{d}\tilde{x} \right)^{1/2} \\
&\leqslant \left(\int_0^1 |f(\tilde{x}) - g(\tilde{x})|^2 \, \mathrm{d}\tilde{x} \right)^{1/2} \\
&< \left(\int_0^1 \epsilon^2 \, \mathrm{d}\tilde{x} \right)^{1/2} = \epsilon,
\end{aligned}$$

and thus the Fourier series of f converges to f in the L^2 norm. ☐

Problem 2

Let $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$, which we can identify with $C^0(S^1, \mathbb{C})$. For $n \in \mathbb{Z}$, recall that

$$\hat{f}(n) = \langle e_n, f \rangle_{L^2} := \int_0^1 f(x) e^{-2\pi i n \tilde{x}} \, \mathrm{d}\tilde{x}$$

is the n^{th} Fourier coefficient of f. Prove that the doubly infinite series of real numbers $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ converges, with limit equal to $\|f\|^2_{L^2}$.

Proof. Recall that $\{e_n\}$ is orthonormal and so are its subsets. If we define $S_n(f) := \sum_{k=-n}^{n} \hat{f}(k)e_k$, then it becomes clear that $S_n(f)$ is the orthogonal projection of f onto $\text{span}\{e_{-n}, \ldots, e_n\}$. Therefore by the previous problem set we immediately know $f - S_n(f)$ is orthogonal to $S_n(f)$. Notice that

$$\begin{aligned}\|f\|_{L^2}^2 = \langle f, f\rangle_{L^2} &= \langle f - S_n(f) + S_n(f), f - S_n(f) + S_n(f)\rangle_{L^2} \\ &= \langle f - S_n(f), f - S_n(f)\rangle_{L^2} + \langle S_n(f), S_n(f)\rangle_{L^2} + \underbrace{2\mathfrak{Re}\,\langle f - S_n(f), f\rangle}_{=0} \\ &= \|f - S_n(f)\|_{L^2}^2 + \|S_n(f)\|_{L^2}^2.\end{aligned}$$

From problem 1, we know that, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ sufficiently large such that $\|f - S_n(f)\|_{L^2}^2 < \epsilon$ whenever $n \geqslant N$. When this happens, the equation above suggests $\|f\|_{L^2}^2 - \epsilon < \|S_n(f)\|_{L^2}^2 < \|f\|_{L^2}^2$. Therefore as $n \to \infty$ we have $\|S_n(f)\|_{L^2}^2 \to \|f\|_{L^2}^2$. On the other hand, by generalized Pythagorean Theorem,

$$\begin{aligned}\|S_n(f)\|_{L^2}^2 = \left\| \sum_{k=-n}^{n} \hat{f}(k)e_k \right\| &= \left\langle \sum_{k=-n}^{n} \hat{f}(k)e_k, \sum_{k=-n}^{n} \hat{f}(k)e_k \right\rangle_{L^2} \\ &= \sum_{j=-n}^{n} \sum_{m=-n}^{n} \hat{f}(j)\hat{f}(m) \underbrace{\langle e_j, e_m\rangle_{L^2}}_{=\delta(j,m)} \\ &= \sum_{j=-n}^{n} \hat{f}(j)^2 = \sum_{j=-n}^{n} |\hat{f}(j)|^2,\end{aligned}$$

and letting $n \to \infty$ gives $\|S_n(f)\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$. Therefore the two must equal, i.e.,

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \|f\|_{L^2}^2. \qquad \text{(Parseval's Identity)}$$ □

Problem 3

Show that if $f \in C^0(S^1, \mathbb{C})$ then

$$\lim_{n \to \pm\infty} \hat{f}(n) = 0.$$

Proof. Since each $|\hat{f}(n)|^2 \geqslant 0$ and $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ converges, do do $\sum_{n=0}^{-\infty} |\hat{f}(n)|^2$ and $\sum_{n=1}^{\infty} |\hat{f}(n)|^2$. Therefore

$$\lim_{n \to \pm\infty} |\hat{f}(n)|^2 = 0 \implies \lim_{n \to \pm\infty} |\hat{f}(n)| = 0.$$ □

Problem 4

Define $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ by setting $f(x) = (1/2 - x)^2$ for $x \in [0, 1]$ and extending f periodically to all or $\mathbb{R}$. Compute the Fourier coefficients $\hat{f}(n)$ for all $n \in \mathbb{Z}$ and write down the corresponding Fourier series as a doubly infinite sum of exponentials. Rewrite the series as a single infinite sum of cosines.

Solution

If $n = 0$, $\hat{f}(n)$ is simply

$$\int_0^1 e^0 (1/2-\theta)^2 \,\mathrm{d}\theta = -\frac{1}{3}(1/2-\theta)^3\Big|_{\theta=0}^{1} = \frac{1}{12}.$$

Before computing $\hat{f}(n)$ for nonzero n's, we first compute the indefinite integral $\int e^{-2\pi i n\theta} f(\theta)\,\mathrm{d}\theta$.

$$\int e^{-2\pi i n\theta}(1/2-\theta)^2\,\mathrm{d}\theta \qquad \begin{bmatrix} u = (1/2-\theta)^2 & \mathrm{d}u = (2\theta-1)\mathrm{d}\theta \\ \mathrm{d}v = e^{-2\pi i n\theta}\mathrm{d}\theta & \begin{aligned} v &= e^{-2\pi i n\theta}/(-2\pi i n) \\ &= ie^{-2\pi i n\theta}/(2\pi n)\end{aligned} \end{bmatrix}$$

$$= \frac{ie^{-2\pi i n\theta}}{2\pi n}(1/2-\theta)^2 - \int \frac{ie^{-2\pi i n\theta}}{2\pi n}(2\theta-1)\,\mathrm{d}\theta.$$

Now we apply integration by parts again to the new integral:

$$\int \frac{ie^{-2\pi i n\theta}}{2\pi n}\,\mathrm{d}\theta = \frac{i}{2\pi n}\int e^{-2\pi i n\theta}(2\theta-1)\,\mathrm{d}\theta \qquad \begin{bmatrix} u = 2\theta-1 & \mathrm{d}u = 2\mathrm{d}\theta \\ \mathrm{d}v = e^{-2\pi i n\theta}\mathrm{d}\theta & v = ie^{-2\pi i n\theta}/(2\pi n)\end{bmatrix}$$

$$\begin{aligned}
&= \frac{i}{2\pi n}\left[\frac{ie^{-2\pi i n\theta}}{2\pi n}(2\theta-1) - \int \frac{2i}{2\pi n}e^{-2\pi i n\theta}\,\mathrm{d}\theta\right] \\
&= \frac{i}{2\pi n}\left[\frac{ie^{-2\pi i n\theta}}{2\pi n}(2\theta-1) - \frac{i}{\pi n}\cdot\frac{ie^{-2\pi i n\theta}}{2\pi n}\right] \\
&= \frac{-e^{-2\pi i n\theta}}{4\pi^2 n^2}(2\theta-1) + \frac{ie^{-2\pi i n\theta}}{4\pi^3 n^3}.
\end{aligned}$$

Therefore,

$$\int e^{-2\pi i n\theta}(1/2-\theta)^2\,\mathrm{d}\theta = \frac{ie^{-2\pi i n\theta}}{2\pi n}(1/2-\theta)^2 + \frac{e^{-2\pi i n\theta}}{4\pi^2 n^2}(2\theta-1) - \frac{ie^{-2\pi i n\theta}}{4\pi^3 n^3}.$$

Evaluating this at $\Big|_{\theta=0}^{1}$, we obtain

$$\begin{aligned}
\hat{f}(n) = \int_0^1 e^{-2\pi i n\theta}(1/2-\theta)^2\,\mathrm{d}\theta &= \frac{ie^{-2\pi i n}}{8\pi n} + \frac{e^{-2\pi i n}}{4\pi^2 n^2} - \frac{ie^{-2\pi i n}}{4\pi^3 n^3} - \frac{i}{8\pi n} + \frac{1}{4\pi^2 n^2} + \frac{i}{4\pi^3 n^3} \\
&= e^{-2\pi i n}\left[\frac{i}{8\pi n} + \frac{1}{4\pi^2 n^2} - \frac{i}{4\pi^3 n^3}\right] - \frac{i}{8\pi n} + \frac{1}{4\pi^2 n^2} + \frac{i}{4\pi^3 n^3}.
\end{aligned}$$

To compute the Fourier series, we compute $\hat{f}(n)e^{2\pi in\theta}$ with $\hat{f}(-n)e^{-2\pi in\theta}$ together:

$$\begin{aligned}\hat{f}(n)e_n + \hat{f}(-n)e_{-n} &= \left[\frac{i}{8\pi n} - \frac{i}{4\pi^3 n^3}\right](e^{2\pi in(\theta-1)} - e^{-2\pi in(\theta-1)}) \\ &\quad + \frac{1}{4\pi^2 n^2}(e^{2\pi in(\theta-1)} + e^{-2\pi in(\theta-1)}) \\ &\quad + \frac{1}{4\pi^2 n^2}(e^{2\pi in\theta} + e^{-2\pi in\theta}) \\ &\quad - \left[\frac{i}{8\pi n} - \frac{i}{4\pi^3 n^3}\right](e^{2\pi in\theta} - e^{-2\pi in\theta}) \\ &= 2i\left[\frac{i}{8\pi n} - \frac{i}{4\pi^3 n^3}\right]\Big[\sin(2\pi n(\theta-1)) - \sin(2\pi n\theta)\Big] \\ &\quad + \frac{2}{4\pi^2 n^2}\Big[\cos(2\pi n(\theta-1)) + \cos(2\pi n\theta)\Big] \\ &= \left[\frac{1}{2\pi^3 n^3} - \frac{1}{4\pi n}\right]\Big[\sin(2\pi n(\theta-1)) - \sin(2\pi n\theta)\Big] \\ &\quad + \frac{2}{4\pi^2 n^2}\Big[\cos(2\pi n(\theta-1)) + \cos(2\pi n\theta)\Big] \\ &= \frac{2}{4\pi^2 n^2}\cdot 2\cos(2\pi n\theta) = \frac{\cos(2\pi n\theta)}{\pi^2 n^2}.\end{aligned}$$

Therefore, the Fourier series is simplified to

$$f(\theta) = \frac{1}{12} + \sum_{n=1}^{\infty}\frac{\cos(2\pi n\theta)}{\pi^2 n^2}.$$

This series is itself 1-periodic, and we are done.

Problem 5

Let $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ and suppose that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ converges. Prove that the Fourier series of f converges uniformly to f. Deduce that the Fourier series of $f(x) = (1/2 - x)^2$ converges uniformly to f.

Proof. By the hint, we consider the Weierstraß M-test. Notice that

$$\left\|\sum_{k=-n}^{n}\hat{f}(k)e_k\right\| \leqslant \sum_{k=-n}^{n}\|\hat{f}(k)e_k\| = \sum_{k=-n}^{n}|\hat{f}(k)|\|e_k\| = \sum_{k=-n}^{n}|\hat{f}(k)|.$$

The one on the RHS is convergent by assumption, so the Fourier series converges absolutely uniformly. By the first problem we know $S_n(f) \to f$ in $\|\cdot\|_{L^2}$. By the uniqueness of limit, it follows that the Fourier series must converge uniformly to f.

As an application of what is shown above, $f \in C^0_{per}(\mathbb{R}, \mathbb{C})$*, should have a convergent series of Fourier coefficients. I must have missed some important details when attempting to simplify the Fourier coefficients. Prof. Manion gave me the reference of the problem after I correctly computed the Fourier series in an overcomplicated way, but the source omitted the integration. I know that here he wanted me to use the fact that* $\sum_{n=-\infty}^{\infty}|\hat{f}(n)| < \infty$ *which leads to uniform convergence of the Fourier series. Unfortunately I cannot achieve such goal using my method.* ☐