HOMEWORK, WEEK 9

This assignment is due Monday, March 22. Handwritten solutions are acceptable but LaTeX solutions are preferred. You must write in full sentences (abbreviations and common mathematical shorthand are fine).

(1) Let U be an open subset of $\mathbb{R}^m$ and let $F : U \to \mathbb{R}^m$ be Lipschitz. Let (a, b) and (a', b') be open intervals of $\mathbb{R}$, each containing t_0. Let $\gamma : (a, b) \to U$ and $\zeta : (a', b') \to U$ be differentiable functions such that $\gamma' = F(\gamma)$, $\zeta' = F(\zeta)$, and $\gamma(t_0) = \zeta(t_0)$. Prove that $\gamma(t) = \zeta(t)$ for all $t \in (a, b) \cap (a', b')$.

See the end of this problem set for an extended remark on this problem.

Hint: This is a stronger version of the uniqueness statement in the Picard–Lindelöf theorem. Let

$$s = \sup\{t \in (a, b) \cap (a', b') \mid \gamma(t') = \zeta(t') \text{ for } t_0 \le t' \le t\}.$$

Assuming $s < \min(b, b')$, try to derive a contradiction as follows. First show that $\gamma(s) = \zeta(s)$ (it's helpful that the set of t where $\gamma(t) = \zeta(t)$ is closed- why is this true?). Then apply Picard's theorem to the functions $\gamma(t + s)$ and $\zeta(t + s)$, which have the same initial value at $t = 0$, and try to derive a contradiction. A parallel argument should show that $\gamma(t) = \zeta(t)$ for all $t \in (\max(a, a'), t_0]$ (you can just say this without writing everything out twice).

(2) Pugh, Exercise 4.35. You should find an explicit formula for the solution, valid on some open neighborhood of zero; this explicit formula will let you answer the rest of the question.

Hint: The equation $x' = x^2$ is separable, meaning that we can rearrange to get all instances of x and its derivatives on the left, and all "unbound" instances of t on the right (e.g. $x' = tx^2$ is also separable, but $x' = e^{tx}$ is not).

To find the solution, don't worry about rigor at first- just use optimistic calculus manipulations to guess the formula, and then prove it's valid afterwards. To find the guess, rewrite the equation as $\frac{x'(t)}{x(t)^2} = 1$, integrate both sides with respect to t, and use substitution on the left. The indefinite integration produces a constant C, and you can determine C using x_0. The resulting formula might have some instances of x_0 in denominators; by multiplying through, you can clear these and get a formula that should work even when $x_0 = 0$.

Now that you have your explicit formula, check rigorously that it's a solution with initial value x_0. Define its domain explicitly (should be an open neighborhood of zero), compute its derivative to show the equation is satisfied (differentiability follows from the formulas you're using), and compute that the initial value is x_0.

Finally, to answer the last question of the problem, let (a, b) be the explicit domain you defined for your solution (the largest possible while remaining connected). You will have $a = -\infty$ and $b < \infty$ or the other way around. In the first case, compute that $x(t) \to \infty$ as $t \to b^-$, so that the solution blows up in finite time. The second case is similar (you don't have to write it out again).

Remark. The statement of the Picard–Lindelöf theorem in the book immediately implies the following, more local version: if $F : U \to \mathbb{R}^m$ satisfies a *local* Lipschitz condition, in the sense that for all $p \in U$, there exists an open neighborhood V of p (contained in U) such that $F|_V$ is Lipschitz, then the conclusions of the theorem hold (just replace U with V everywhere in the proof; the conclusions are local and are thus unchanged). It is also true (but requires an additional argument) that if $U = \mathbb{R}^m$ and $F : \mathbb{R}^m \to \mathbb{R}^m$ has a (global) Lipschitz constant, then solutions to $\gamma' = F(\gamma)$ exist for all time. However, even if $U = \mathbb{R}^m$ (as in the previous problem), if F is only locally Lipschitz (e.g. $F(x) = x^2$ on $\mathbb{R}^1$), then solutions to $\gamma' = F(\gamma)$ might blow up in finite time. One can show the solutions (defined on their maximal interval of existence) must leave every compact subset K of U (e.g. $K =$ a rectangle in $\mathbb{R}^2$), but they might "leave to the top or bottom" instead of "leaving to the right or left."

(3) An important operation in Fourier analysis is *convolution*, e.g. of periodic functions $f, g \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$. This operation can be understood abstractly as follows: the map $f \mapsto (\hat{f}(n))_{n=-\infty}^{\infty}$ is an injective $\mathbb{C}$-linear map from $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ to $\ell^2_{\mathbb{C}}$ that preserves inner products (exercise if you want: use L^2 convergence to prove the map is injective). However, both $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ and $\ell^2_{\mathbb{C}}$ have "function multiplication" ($f, g \mapsto fg$ in the first case, $(a_n), (b_n) \mapsto (a_n b_n)$ in the second case. Our map $f \mapsto (\hat{f}(n))_{n=-\infty}^{\infty}$ does *not* preserve function multiplication; we do not have $\hat{fg}(n) = \hat{f}(n)\hat{g}(n)$.

However, this is a feature, not a bug: we can use the map $f \mapsto (\hat{f}(n))_{n=-\infty}^{\infty}$ to define a new type of multiplication for functions $f, g \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$. Namely, given f and g, we can take their sequences $(\hat{f}(n))$, $(\hat{g}(n))$ of Fourier coefficients, multiply these to get $(\hat{f}(n)\hat{g}(n))_{n=-\infty}^{\infty}$, and reconstruct a new function $f * g(x) := \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)e^{2\pi i n x}$ (at least once we check the details). Similarly, given sequences (a_n) and (b_n), heuristically we can construct $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$ and $g(x) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi i n x}$, multiply to get fg, and then define the convolution $(a_n) * (b_n)$ to be $(\widehat{fg}(n))$. (In general, one can define convolution for appropriate functions on locally compact topological abelian groups G, by using the Fourier transform relating functions on G to functions on its *Pontryagin dual* $\hat{G}$).

It turns out that this roundabout way of defining convolution is equivalent to a more direct way:

Definition. For $f, g \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$, define $f * g \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ by

$$(f * g)(x) = \int_0^1 f(y)g(x-y)dy.$$

For $(a_n)_{n=-\infty}^{\infty}$, $(b_n)_{n=-\infty}^{\infty} \in \ell^2_{\mathbb{C}}$ define

$$(a_n) * (b_n) = \left(\sum_{k=-\infty}^{\infty} a_k b_{n-k} \right)_{n=-\infty}^{\infty}.$$

You may assume that both types of convolution operations give well-defined elements of $\mathbb{C}^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ and of $\ell^2_{\mathbb{C}}$ respectively. It's a good integration exercise to check that $f * g = g * f$.

Problem. Given $(a_n)_{n=-\infty}^{\infty}$, $(b_n)_{n=-\infty}^{\infty}$ in $\ell^2_{\mathbb{C}}$, assume that the Fourier series

- $\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\theta}$,
- $\sum_{n=-\infty}^{\infty} b_n e^{2\pi i n\theta}$,
- $\sum_{n=-\infty}^{\infty} a_n b_n e^{2\pi i n\theta}$

converge absolutely (for any θ), and thus uniformly to functions f, g and h in $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$. Prove that for $x \in \mathbb{R}$, we have $h(x) = \int_0^1 f(y)g(x-y)dy$. You may assume without proof that a uniformly convergent double sum $\sum_n \sum_m$ can be interchanged with a Riemann integral $\int_0^1 \dots dx$ (even if both indices n, m are doubly infinite); this follows e.g. from generalizing HW 2 (problem 2abd) to interchange limits of nets with limits of nets given appropriate uniform convergence.

Hint: Write out $\int_0^1 f(y)g(x-y)dy$ as the integral of a uniformly convergent double sum and interchange the sum with the integral. Eventually you can bring the integral inside until you just have the integral of an exponential and you can use the L^2 orthogonality of the functions $(e_n)_{n\in\mathbb{Z}}$ proved in HW 6/7.

(4) In the previous problem, we showed that multiplication of Fourier coefficients corresponds to convolution of functions. Now we consider the "opposite direction:" multiplication of functions corresponds to convolution of Fourier coefficients. The argument should be parallel, but the parallel proof requires higher technology (Dirac delta functions rather than Kronecker delta functions, which are best formalized in terms of distribution theory). Thus, in this problem, you should ignore rigor and just calculate heuristically (it's good not to forget how to do this).

Problem. For $f, g \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ and $n \in \mathbb{Z}$, show using heuristic calculations that $\widehat{fg}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(n-k)$. You may freely interchange integrals and series. You may also assume the formula $\sum_{n=-\infty}^{\infty} e^{2\pi i n(y-x)} = \delta_x(y)$, where δ_x denotes the Dirac delta function at x (zero away from x and "infinitely spiked" at x). The "function" δ_x is characterized by how it integrates against functions $F(y)$: we have $\int_0^1 F(y)\delta_x(y)dy = F(x)$ basically by definition (once the technology is set up).

Hint: Expand out the sum $\sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(n-k)$ using the integral definition of the Fourier coefficients, then bring the sum inside the integrals as far as possible until it can be replaced by a Dirac delta function. Use the delta function to change the double integral into a single integral.

Remark. Heuristically, the Fourier coefficients of δ_0 should each be 1, since

$$\int_0^1 \delta_0(x) e^{-frm-e\pi i n x} dx = e^{-2\pi i n 0} = 1.$$

Thus, "reconstructing δ_0 in terms of its Fourier series," we should have $\delta_0(y) = \sum_{n=-\infty}^{\infty} e^{2\pi i n y}$; shifting variables, we should have $\delta_x(y) = \sum_{n=-\infty}^{\infty} e^{2\pi i n(y-x)}$.

Remark. Given the first problem, we can now see rigorously (without using theorems from complex analysis) that our two definitions of $\exp(x)$ (inverse of $\log(x) = \int_1^x 1/t dt$ and power series) agree on all of $\mathbb{R}$, since they both solve the initial value problem $y' = y, y(0) = 1$ on all of $\mathbb{R}$. Yet another way to prove this fact is using results about Taylor series from Section 4.6.

Suppose one defines $\arcsin(x) = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$ as discussed previously. By the fundamental theorem of calculus, this arcsine function is differentiable on $(-1, 1)$, and one can show it maps $[-1, 1]$ bijectively to $[-\pi/2, \pi/2]$ (where π is defined to be half the circumference of the unit circle). By the 1d inverse function theorem (we will prove the n-dimensional version later), the inverse function $\sin : [-\pi/2, \pi/2] \to [-1, 1]$ is differentiable on $(-\pi/2, \pi/2)$ with derivative $\sin' = \sqrt{1 - \sin^2}$, and one can show that sin is right/left differentiable at the endpoints $\{-\pi/2, \pi/2\}$ with derivative zero. Differentiating again, one gets $\sin'' = -\sin$. Since $\arcsin(0) = 0$, we have $\sin(0) = 0$ and thus $\sin'(0) = \sqrt{1 - \sin^2(0)} = 1$.

One can check that the same second-order initial value problem is solved by the power series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ (the Taylor series of sin at zero, computable from $\sin'' = -\sin$ and the initial-value data), which has radius of convergence ∞. By the global uniqueness result proved in the first problem, $\sin(x)$ agrees with the power series on $(-\pi/2, \pi/2)$, so we can use the power series to define sin (and thus $\sin'$) on all of $\mathbb{R}$. An analogous argument shows that $\sin'$ agrees with the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

Remark. As previously discussed, using the series definitions, one can deduce Euler's relation $e^{ix} = \sin'(x) + i \sin(x)$; then one can use the addition formula $e^{x+y} = e^x e^y$ to deduce the addition formula $\sin(x + y) = \sin(x) \sin'(y) + \sin'(x) \sin(y)$ for the sine function.

The definition of arcsine in the previous remark is based on an arc-length integral for a circle; it is natural to ask what happens if the circle is replaced by a different curve such as an ellipse. The resulting elliptic integrals typically can't be evaluated in terms of elementary functions (the same would be true for the integrals $\int_1^x 1/t dt$, $\int_0^x 1/\sqrt{1 - t^2} dt$ above except that exp, sin, etc. and their inverses are considered "elementary" for historical reasons). Instead, one uses them to define "generalized inverse trigonometric functions" and their inverses, "generalized trigonometric functions" or "elliptic functions." These have addition formulas too (looking like $\wp(x + y) = \frac{1}{4} \left(\frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)} \right)^2 - \wp(x) - \wp(y)$ in modern notation), related to group laws on elliptic curves and thereby to many deep questions in number theory. A major discovery was made by Abel in 1827-8: when extended to the complex plane, these elliptic functions $\wp(z)$ have *two* independent directions of periodicity, so their natural domain is a complex torus. The addition formula for elliptic functions encodes the fact that these functions send the usual group operation on the complex torus (addition modulo the period lattice) to a geometrically-defined group operation on an algebraic curve (an elliptic curve) serving as the natural domain for the elliptic integral in question. In number theory, much work is devoted to understanding the subgroup consisting of rational points on the elliptic curve.