MATH 425b Homework 9

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March 22, 2021

Problem 1

Let U be an open subset of $\mathbb{R}^m$ and let $F : U \to \mathbb{R}^m$ be Lipschitz. Let (a, b) and (a', b') be open intervals of $\mathbb{R}$, each containing t_0. Let $\gamma : (a, b) \to U$ and $\zeta : (a', b') \to U$ be differentiable functions with $\gamma' = F(\gamma)$, $\zeta' = F(\zeta)$, and $\gamma(t_0) = \zeta(t_0)$. Show that $\gamma(t) = \zeta(t)$ for all $t \in (a, b) \cap (a', b')$.

Proof. We first show that $\gamma(t) = \zeta(t)$ for all $t \in [t_0, b) \cap [t_0, b')$. Per the hint, define

$$s := \sup \mathcal{S} := \sup\{t \in (a, b) \cap (a', b') \mid \gamma(t') = \zeta(t') \text{ for } t_0 \leqslant t' \leqslant t\}.$$

$\mathcal{S}$ is clearly nonempty as $t_0 \in \mathcal{S}$ and it is also bounded from above by $\min(b, b')$. Therefore $s = \sup \mathcal{S}$ is well-defined. We want to show that $s = \min(b, b')$, so suppose for contradiction that $s < \min(b, b')$. By the definition of supremum there exists a strictly increasing sequence $\{t_n\}$ that converges to s. Since γ and ζ are continuous, $\gamma(t_n) \to \gamma(s)$ and $\zeta(t_n) \to \zeta(s)$. Notice that $\{\gamma(t_n)\}$ and $\{\zeta(t_n)\}$ are identical, so by the uniqueness of limits $\gamma(s) = \zeta(s)$. By our assumption $s \in (a, b) \cap (a', b')$, and the Picard-Lindelöf theorem says that γ and ζ agree on some open neighborhood of s. This means γ and ζ agree on $[s, s + \epsilon$ for some $\epsilon > 0$, contradicting the assumption that $s = \sup \mathcal{S}$. Therefore $s = \min(b, b')$. An analogous argument can show that $\gamma(t) = \zeta(t)$ for all $t \in (a, t_0] \cap (a', t_0]$ by defining

$$r := \inf \mathcal{R} := \inf\{t \in (a, b) \cap (a', b') \mid \gamma(t') = \zeta(t') \text{ for all } t \leqslant t' \leqslant t_0\}.$$

Thus γ and ζ agree on all of $(a, b) \cap (a', b')$. □

Problem 2

(Pugh, Ex.4.35.) Consider the ODE $x' = x^2$ on $\mathbb{R}$. Find the solution of the ODE with initial condition x_0. Are the solutions to this ODE defined for all time or do they escape to infinity in finite time?

Solution

We first compute the general solution by separation of variables:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 \implies \frac{\mathrm{d}x}{x^2} = \mathrm{d}t \implies \int_0^t \frac{\mathrm{d}x}{x^2} = \int_0^t \mathrm{d}t \implies -\frac{1}{x} = t + C.$$

From the initial condition that $x(t_0) = x_0$ we get $-1/x_0 = t_0 + C \implies C = -t_0 - 1/x_0$. Therefore,

$$-\frac{1}{x} = t - t_0 - \frac{1}{x_0} \implies x(t) = -\frac{x_0}{tx_0 - t_0x_0 - 1}.$$

If the initial condition is given at $t_0 = 0$ this simply reduces to

$$x(t) = \frac{x_0}{1 - tx_0}.$$

Note that $x(t)$ is indeed well-defined at $x(0)$. In fact, we can define its domain to be $(-1/x_0, 1/x_0)$, on which

$$x' = -x_0(1 - tx_0)^2(-x_0) = x^2.$$

This solution blows up in finite time; in particular as $t \uparrow 1/x_0$ or $t \downarrow -1/x_0$ the denominator $(1 - tx_0) \to 0$ and so $x(t) \to \infty$.

Problem 3

Given doubly infinite sequences $\{a_n\}, \{b_n\} \in \ell^2_{\mathbb{C}}$, assume that the Fourier series

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\theta}, \sum_{n=-\infty}^{\infty} b_n e^{2\pi i n\theta}, \text{ and } \sum_{n=-\infty}^{\infty} a_n b_n e^{2\pi i n\theta}$$

converge absolutely (for any θ) and thus uniformly to functions f, g, and h in $C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$. Prove that for $x \in \mathbb{R}$ we have

$$h(x) = \int_0^1 f(y)g(x - y)\,\mathrm{d}y.$$

Proof. Notice that

$$f(y) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n y} \text{ and } g(\theta - y) = \sum_{m=-\infty}^{\infty} b_n e^{2\pi i n(\theta - y)}.$$

What we want to show is that the convolution is the function to which $\sum_{n=-\infty}^{\infty} a_n b_n e^{2\pi i n\theta}$ converges uniformly to for any θ. Indeed,

$$\begin{aligned}
h(\theta) &= \int_0^1 f(y)g(\theta - y)\,\mathrm{d}y \\
&= \int_0^1 \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n y} \sum_{m=-\infty}^{\infty} b_m e^{2\pi i m(\theta - y)}\,\mathrm{d}y \\
&= \int_0^1 \sum_{m,n=-\infty}^{\infty} a_n b_n \exp[2\pi i(ny + m(\theta - y))]\,\mathrm{d}y \\
&= \sum_{m,n=-\infty}^{\infty} a_n b_m \int_0^1 \exp[2\pi i(ny + m(\theta - y))]\,\mathrm{d}y.
\end{aligned}$$

For cases where $m = n$, we have

$$\exp(2\pi i(ny + m\theta - my)) = \exp(2\pi i m\theta)$$

so

$$a_n b_m = a_n b_n \int_0^1 e^{2\pi i n\theta}\,\mathrm{d}y = a_n b_n e^{2\pi i n\theta}.$$

Otherwise (if $m \neq n$) we have $ny + m(\theta - y) = m\theta - (m-n)y$, and so

$$\begin{aligned}\exp(2\pi i(m\theta - (m-n)y)) &= \exp(2\pi i m\theta)\exp(2\pi i(m-n)y) \\ &= \exp(2\pi i(m-n)y).\end{aligned}$$

Integrating this gives

$$a_n b_m \int_0^1 e^{2\pi i(m-n)y}\, \mathrm{d}y = \frac{a_n b_m}{2\pi i(m-n)}\left[e^{2\pi i(m-n)y}\right]_{y=0}^1 = 0.$$

Therefore, the double sum $\sum_{m,n=-\infty}^{\infty}$ can be reduced to $\sum_{n=-\infty}^{\infty}$ only, and

$$h(\theta) = \sum_{n=-\infty}^{\infty} a_n b_n e^{2\pi i n\theta},$$

as desired. This shows that indeed the doubly-infinite series converge to $f * g$, i.e., multiplication of Fourier coefficients corresponds to the convolution of functions.

□

Problem 4

For $f, g \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ and $n \in \mathbb{Z}$, show using heuristic calculations that $\widehat{fg}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(n-k)$. You may freely interchange integrals and series; you may assume the formula $\sum_{k=-\infty}^{\infty} e^{2\pi i k(y-x)} = \delta_x(y)$ where δ_x denotes the *Dirac delta function* at x. We have $\int_0^1 F(y)\delta_x(y)\,\mathrm{d}y = F(x)$.

Proof. Let the brute force computation begin!!

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(n-k) &= \sum_{k=-\infty}^{\infty} \int_0^1 f(x)e^{-2\pi i k x}\,\mathrm{d}x \int_0^1 g(y)e^{-2\pi i(n-k)y}\,\mathrm{d}y \\ &= \sum_{k=-\infty}^{\infty} \int_0^1 f(x)\int_0^1 g(y)\exp\left[-2\pi i(kx + (n-k)y)\right]\,\mathrm{d}y\mathrm{d}x \\ &= \sum_{k=-\infty}^{\infty} \int_0^1 f(x)\int_0^1 g(y)\exp\left[-2\pi i n y\right]\exp\left[2\pi i k(y-x)\right]\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 f(x)\int_0^1 g(y)e^{-2\pi i n y}\sum_{k=-\infty}^{\infty}\exp\left[2\pi i k(y-x)\right]\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 f(x)\int_0^1 g(y)e^{-2\pi i n y}\delta_x(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 f(x)g(x)e^{-2\pi i n x}\,\mathrm{d}x \\ &= \widehat{fg}(n).\end{aligned}$$

□