Problem 1

(a) Let X be a set and (Y, d) a metric space. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $X \to Y$. Define what it means for $\{f_n\}_{n=1}^{\infty}$ to converge uniformly to some function $f : X \to Y$. Also, given a sequence of functions $\{g_n\}_{n=1}^{\infty}$ from X to a formed vector space $(V, \|\cdot\|)$, define what it means for the series $\sum_{n=1}^{\infty} g_n$ to converge uniformly.

(b) Recall that one way to characterize convergence of doubly-infinite series $\sum_{n=-\infty}^{\infty} g_n$ is by requiring that $\sum_{n=0}^{\infty} g_n$ converges and that $\sum_{n=1}^{\infty} g_{-n}$ converges (pointwise, uniformly, etc.). Show that the doubly infinite series

$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2}$$

converges uniformly for $t \in [a, b]$ assuming $0 < a < b$.

(c) For a fixed $t > 0$, consider the function $g_t(x) = e^{-\pi t x^2}$, a Gaussian with some normalization. You may assume without proof that the Fourier transform $\hat{g}_t(\xi)$ exists and is equal to $e^{\pi \xi^2/t}/\sqrt{t}$. Show that

$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t}.$$

[Hint: Poisson summation formula.]

Solution.

(a) $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f if the following criterion is met:

For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n \geqslant N$, $d(f_n(x), f(x)) < \epsilon$ for all x.

Likewise, $\sum_{n=1}^{\infty} g_n$ converges uniformly if the sequence of its partial sums converges uniformly, i.e., if

$$\{h_k\}_{k=1}^{\infty} \text{ defined by } h_k := \sum_{i=1}^{k} g_n$$

converges uniformly.

(b) *Proof.* Notice that if $f(x) = e^{-\pi x n^2}$ for $x \in [a, b]$ as defined in the problem, then $\|f\|_{\infty} = f(a)$ (because this exponential function is strictly decreasing).

Consider the series of constants $\sum_{n=1}^{\infty} e^{-\pi a n^2}$. Since we only focus on $t \in [a, b]$, each term $e^{-\pi t n^2}$ is bounded by $e^{-\pi a n^2}$. Therefore $\sum_{n=1}^{\infty} e^{-\pi t n^2} \leqslant \sum_{n=1}^{\infty} e^{-\pi a n^2}$. Also, since exponentials are always positive, if we try to bound the series, it does not hurt to add some extra terms:

$$\sum_{n=0}^{\infty} e^{-\pi t n^2} \leqslant \sum_{n=0}^{\infty} e^{-\pi a n^2} = \sum_{k \text{ square}} e^{-\pi a k} < \sum_{k=0}^{\infty} e^{-\pi a k}.$$

The last one is a geometric series with exponential growth rate $1/e < 1$ so it converges to a finite number.

Therefore so is the first one, and clearly $\sum_{i=1}^{\infty} e^{-\pi t(-n)^2} = \sum_{i=1}^{\infty} e^{-\pi t n^2} < \sum_{k=1}^{\infty} e^{-\pi a k}$.

We just bounded both singly-infinite series of functions (with respect to $\|\cdot\|_\infty$) by a convergent series of constants and so we can invoke the Weierstraß M-test and conclude that $\sum_{n=0}^{\infty} e^{-\pi t n^2}$ and $\sum_{n=1}^{\infty} e^{-\pi t(-n)^2}$ converges uniformly on $[a,b]$. Hence $\sum_{n=-\infty}^{\infty} e^{-\pi t n^2}$ also converges uniformly. □

(c) Poisson summation formula states that $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$, and this is precisely what the equation is:

$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = \sum_{\xi=-\infty}^{\infty} \frac{\exp(-\pi\xi^2/t)}{\sqrt{t}} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t}. \qquad \square$$

Problem 2

(a) Let V, W be normed vector spaces and let $T : V \to W$ be a linear transformation. Define the operator norm $\|T\|_{\text{op}}$ of T.

(b) Let V, W, Z be normed vector spaces and let $S : V \to W$ and $T : W \to Z$ be linear transformations. Assume that both $\|T\|_{\text{op}}$ and $\|S\|_{\text{op}}$ are finite. Show that

$$\|T \circ S\|_{\text{op}} \leqslant \|T\|_{\text{op}} \|S\|_{\text{op}}.$$

Solution.

(a) $\|T\|_{\text{op}} = \inf\{L > 0 : \|T(v)\| \leqslant L(v) \text{ for all } v \in V\} = \sup_{\|v\|=1} \|T(v)\| = \sup_{\|v\|\leqslant 1} \|T(v)\| = \sup_{v \neq 0} \frac{\|T(v)\|}{\|v\|}$.

(b) *Proof.* The claim is trivial when one of them has 0 operator norm. For the nontrivial case, we will need to use the fact that supremum of product $\leqslant$ product of supremum:

$$\begin{aligned} \|T \circ S\|_{\text{op}} &= \sup_{\|v\|\neq 0} \frac{\|TS(v)\|}{\|v\|} = \sup_{\substack{\|v\|\neq 0 \\ S(v)\neq 0}} \frac{\|TS(v)\|}{\|v\|} = \sup_{\substack{\|v\|\neq 0 \\ S(v)\neq 0}} \frac{\|TS(v)\|}{\|S(v)\|} \cdot \frac{\|S(v)\|}{\|v\|} \\ &\leqslant \sup_{\substack{\|v\|\neq 0 \\ S(v)\neq 0}} \frac{\|TS(v)\|}{\|S(v)\|} \cdot \sup_{\substack{\|v\|\neq 0 \\ S(v)\neq 0}} \frac{\|S(v)\|}{\|v\|} = \sup_{S(v)\neq 0} \frac{\|TS(v)\|}{\|S(v)\|} \|S\|_{\text{op}} \leqslant \|T\|_{\text{op}} \|S\|_{\text{op}}. \qquad \square \end{aligned}$$

Problem 3

(a) Let U be an open subset of $\mathbb{R}^n$ and let F be a function from U to $\mathbb{R}^m$. Define what it means for F to be differentiable at $p \in U$ with total derivative given by a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

(b) (Second option) show that if such T exists then it is unique.

Solution.

(a) We say F is differentiable at p with total derivative T if

$$\lim_{v \to 0} \frac{F(p+v) - F(p) - T(v)}{\|v\|} = 0.$$

(b) *Proof.* If T and S are distinct total derivatives of F at p, then substituting S and T into the definition above and subtracting yields

$$\lim_{v\to 0}\frac{S(v)-T(v)}{\|v\|}=\lim_{v\to 0}\frac{(S-T)(v)}{\|v\|}=0.$$

Now suppose for contradiction that $S-T$ is not the zero linear transformation, i.e., there exists v_0 satisfying $(S-T)(v_0)\neq 0$. We consider cv_0 where $c\in\mathbb{R}^+$. Letting $c\to 0$, we have $cv_0\to 0$, and so

$$\lim_{c\to 0}\frac{(S-T)(cv_0)}{\|cv_0\|}=\lim_{c\to 0}\frac{(S-T)(v_0)}{\|v_0\|}\neq 0,$$

contradiction. Hence $S-T$ must be the zero transformation, i.e., $S=T$. Hence the uniqueness. □

Problem 4

(Second option) prove that C^1 functions are differentiable: if $U\subset\mathbb{R}^n$ is open and $F:U\to\mathbb{R}^m$ a C^1 function on U, then F is differentiable at all $p\in U$ with total derivative at p having the standard-basis matrix

$$(\mathcal{J}F)_p=\begin{bmatrix}\frac{\partial F_1}{\partial x_1}(p) & \cdots & \frac{\partial F_1}{\partial x_n}(p)\\ \vdots & \ddots & \cdots\\ \frac{\partial F_m}{\partial x_1}(p) & \cdots & \frac{\partial F_m}{\partial x_n}(p)\end{bmatrix}.$$

Proof. Let $\mathcal{J}:=(\mathcal{J}F)_p$ and let $R(v):=F(p+v)-F(p)-\mathcal{J}v$. We want to show that $R(v)/\|v\|\to 0$ as $v\to 0$. For $1\leqslant i\leqslant m$, let $R_i(v)$ be the i^{th} coordinate of $R(v)$ and likewise for $F_i(v)$. By definition

$$R_i(v)=F_i(p+v)-F_i(p)-\begin{bmatrix}\frac{\partial F_i}{\partial x_1}(p) & \cdots & \frac{\partial F_i}{\partial x_n}(p)\end{bmatrix}v.$$

It suffices to show that for each i, $R_i(v)/\|v\|\to 0$ as $v\to 0$.
Now let $\epsilon>0$ be given. We choose $\delta>0$ satisfying the following:

(1) If $\|v\|<\delta$ then $p+v\in U$ (possible because U is open) and

(2) $\left|\frac{\partial F_i}{\partial x_j}(p+v)-\frac{\partial F_i}{\partial x_j}(p)\right|<\frac{\epsilon}{n}$ for $1\leqslant j\leqslant n$ (possible because $\frac{\partial F_i}{\partial x_j}$ is continuous).

We claim that this δ satisfies the $\epsilon-\delta$ condition, i.e., if $\|v\|<\delta$ then $|R(v)|/\|v\|<\epsilon$.
To see this, we first rewrite $v=\sum_{j=1}^n v_je_j$ (where e_j's are the standard basis of $\mathbb{R}^n$) and rewrite $R_i(v)$ in the form of a telescoping sum:

$$\begin{aligned}R_i(v)&=F_i(p+\sum_{i=1}^n v_je_j)-F_i(p)-\sum_{j=1}^n v_j\frac{\partial F_i}{\partial x_j}(p)\\ &=F_i(p+\sum_{j=1}^n v_je_j)-F_i(p+\sum_{j=1}^{n-1} v_je_j)-v_n\frac{\partial F_i}{\partial x_n}(p)\\ &\quad+F_i(p+\sum_{j=1}^{n-1} v_je_j)-F_i(p+\sum_{j=1}^{n-2} v_je_j)-v_{n-1}\frac{\partial F_i}{\partial v_{n-1}}(p)\\ &\quad+\cdots+F_i(p+v_1e_1)-F_i(p)-v_1\frac{\partial F_i}{\partial v_1}(p).\end{aligned}$$

Clearly there are n lines in total, and we will show that each line $< \epsilon/n$ using the MVT. For the $(n+1-k)^{\text{th}}$ line (k^{th} counting from bottom), since

$$g : t \mapsto F_i(p + \sum_{j=1}^{k-1} v_j e_j + te_k)$$

is differentiable on $[0, v_k]$ (because $\partial F/\partial x_j$ exists on U), it's well-defined to compute its derivative

$$g'(t) = \frac{\partial F_i}{\partial x_k}(p + \sum_{j=1}^{k-1} v_j e_j + te_k).$$

Therefore $g(v_k) - g(0) = g'(\theta)(v_k - 0)$ for some $\theta \in [0, v_k]$, i.e.,

$$F_i(p + \sum_{j=1}^{k} v_j e_j) - F_i(p + \sum_{j=1}^{k-1} v_j e_j) = v_k \frac{\partial F_i}{\partial x_k}(p + \sum_{j=1}^{k-1} v_j e_j + \theta e_k).$$

Finally, since $\|(v_1, \ldots, v_{k-1}, \theta, 0, \ldots)\| \leqslant \|(v_1, \ldots, v_n)\| = \|v\|$, we have

$$\begin{aligned} \|(n+1-k)^{\text{th}} \text{ line}\| &= \left\| v_k \frac{\partial F_i}{\partial x_k}(p + \sum_{j=1}^{k-1} v_j e_j + \theta e_k) - v_k \frac{\partial F_i}{\partial x_k}(p) \right\| \\ &= |v_k| \left\| \frac{\partial F_i}{\partial x_k}(p + \sum_{j=1}^{k-1} v_j e_j + \theta e_k \right\| - \frac{\partial F_i}{\partial x_k}(p) \right\| \leqslant \frac{|v_k|\epsilon}{n}, \end{aligned}$$

and since $|v_j| \leqslant \|v\|$ for all j,

$$\frac{R_i(v)}{\|v\|} \leqslant \sum_{j=1}^{n} \frac{|v_j|\epsilon}{n\|v\|} \leqslant \sum_{j=1}^{n} \frac{\epsilon}{n} = \epsilon.$$

This proves the claim. □

Problem 5

Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$F(x, y, z) = (4x^2 + 4y^2 - 4z^2, x^2 + y^2 + z^2).$$

(a) Compute the Jacobian matrix of F.

(b) Consider the level set $F^{-1}(1, 1)$. Show that for all points (x, y, z) in this level set the Jacobian matrix F has maximal rank.

Solution.

(a) By definition,

$$(\mathcal{J}F)_{(x,y,z)} = \begin{bmatrix} 8x & 8y & -8z \\ 2x & 2y & 2z \end{bmatrix}.$$

(b) *Proof.* We prove by contradiction. Suppose $\mathcal{J}F$ at some $(x, y, z) \in F^{-1}(1, 1)$ is not of maximal rank. Then it's of rank at most 1. In particular, the following two matrices would be singular:

$$\begin{bmatrix} 8x & 8y \\ 2x & 2y \end{bmatrix} \quad \begin{bmatrix} 8y & -8z \\ 2y & 2z \end{bmatrix}.$$

The first one is, of course, always singular, but if the second one is singular then $z = 0$ (multiply second row by 4 and obtain $8z = -8z$). Then

$$4x^2 + 4y^2 = 1 \qquad x^2 + y^2 = 1,$$

clearly a contradiction. This finishes the proof. □

Problem 6

(1) Let $\beta := (x^2y + z)\mathrm{d}x + (xyz)\mathrm{d}y + (x + yz^2)\mathrm{d}z$, a 1-form on $\mathbb{R}^3$. Compute $\mathrm{d}\beta$.

(2) Let $P(x), Q(x)$ be smooth functions from $\mathbb{R}$ to $\mathbb{R}$. Consider the differential 1-form

$$\alpha := (P(x)y - Q(x))\mathrm{d}x + \mathrm{d}y$$

on $\mathbb{R}^2$. Let

$$\mu(x) := \exp \int_0^x P(t)\,\mathrm{d}t.$$

Prove that the 1-form

$$\mu\alpha = \mu(x)(P(x)y - Q(x))\mathrm{d}x + \mu(x)\mathrm{d}y$$

satisfies $\mathrm{d}(\mu\alpha) = 0$.

Solution.

(a)
$$\begin{aligned}\mathrm{d}\beta &= \left(2xy\mathrm{d}x + x^2\mathrm{d}y + \mathrm{d}z\right) \wedge \mathrm{d}x + (yz\mathrm{d}x + xz\mathrm{d}y + xy\mathrm{d}z) \wedge \mathrm{d}y + (\mathrm{d}x + z^2\mathrm{d}y + 2yz\mathrm{d}z) \wedge \mathrm{d}z \\ &= (yz - x^2)\mathrm{d}x \wedge \mathrm{d}y + (z^2 - xy)\mathrm{d}y \wedge \mathrm{d}z.\end{aligned}$$

(b) We will compute $\mathrm{d}(\mu\alpha)$ by brute force. Note that the x-partial of $\mu(x)(P(x)y - Q(x))$ does not matter because eventually it vanishes with $\mathrm{d}x \wedge \mathrm{d}x$. Meanwhile,

$$\frac{\partial}{\partial y}\left[\mu(x)(P(x)y - Q(x))\right] = \mu(x)P(x).$$

Therefore,

$$\begin{aligned}\mathrm{d}(\mu\alpha) &= \mu(x)P(x)\mathrm{d}y \wedge \mathrm{d}x + \mu'(x)\mathrm{d}x \wedge \mathrm{d}y \\ [\text{chain rule}] &= \mu(x)P(x)\mathrm{d}y \wedge \mathrm{d}x + \mu(x)P(x)\mathrm{d}x \wedge \mathrm{d}y = 0.\end{aligned}$$

□

Problem 7

Prove the Picard-Lindelöf Theorem: if $U \subset \mathbb{R}^m$ is open and $F : U \to \mathbb{R}^m$ is locally Lipschitz, then, given $p \in U$, there exists a locally unique solution to the IVP $\gamma'(t) = F(\gamma(t))$ defined on $(a, b) \in \mathbb{R}$ with $x(t_0) = y_0$.

Proof. For convenience let $t_0 = 0$ (the generic case can be obtained via the integral equation once the case $t_0 = 0$ is proven). WLOG assume F is Lipschitz with constant L on all of U. Pick $r > 0$ such that $N := \overline{B(y_0, r)} \subset U$. Since N is closed and bounded in $\mathbb{R}^m$, it is compact, on which the continuous image $F(N)$ is also compact. Hence there exists $M \in \mathbb{R}$ such that $\|F(x)\| \leqslant M$ for all $x \in N$.

Now pick $\tau > 0$ sufficiently small such that $\tau < \min(r/M, 1/L)$. We claim that

(1) there exists $\gamma : (-\tau, \tau) \to N$ differentiable with $\gamma'(t) = F(\gamma(t))$, and

(2) such γ is unique.

Notice that solving the IVP $\gamma'(t) = F(\gamma(t)), \gamma(0) = y_0$ is equivalent to solving

$$\gamma(t) = \gamma(0 + \int_0^t F(\gamma(s))\, \mathrm{d}s.$$

Since the space $(C^0([-\tau, \tau], N), \mathrm{d}_{\sup})$ is Banach, if we can show that

$$(\Phi(\gamma))(t) := y_0 + \int_0^t F(\gamma(s))\, \mathrm{d}s$$

is a contraction, then by the Banach contraction mapping theorem, there exists a fixed point which would solve our IVP. Clearly $\Phi(\gamma)$ is continuous, and for $t \in [-\tau, \tau]$,

$$\|\Phi(\gamma)(t) - y_0\| = \left\| \int_0^t F(\gamma(s))\, \mathrm{d}s \right\| \leqslant M|t-0| \leqslant M\tau = M \cdot min(r/M, 1/L) \leqslant r,$$

so indeed $\Phi(\gamma)(t)$ is always an element of $C^0([-\tau, \tau], N)$. Now we show that Φ is actually a contraction with constant $\tau L < 1$. If $\gamma, \sigma \in C^0([-\tau, \tau], N)$, then

$$\begin{aligned} d(\Phi(\gamma), \Phi(\sigma)) &= \sup_{t \in [-\tau, \tau]} \left\| y_0 + \int_0^t F(\gamma(s))\, \mathrm{d}s - y_0 - \int_0^t F(\sigma(s))\, \mathrm{d}s \right\| \\ &= \sup_{t \in [-\tau, \tau]} \left\| \int_0^t F(\gamma(s)) - F(\sigma(s))\, \mathrm{d}s \right\| \\ &\leqslant \sup_{t \in [-\tau, \tau]} |\, t \,| \cdot \sup_{s \in [\tau, \tau]} \|F(\gamma(s)) - F(\sigma(s))\| \\ &\leqslant \tau \cdot L \sup_{s \in [-\tau, \tau]} \|\gamma(s) - \sigma(s)\| = \tau L \cdot \mathrm{d}_{\sup}(\gamma, \sigma). \end{aligned}$$

Now we can invoke the Banach contraction mapping theorem and conclude that there exists a solution to the IVP. Uniqueness follows from one of our HWs, in which we've shown that if $\gamma : (a, b) \to U$ and $\sigma : (a', b') \to U$ are two solutions to the IVP then γ and σ must agree on $(a, b) \cap (a', b')$. Local uniqueness still holds. □