Problem: 7(a)

Let $c > 0$. For any complex number s, show that the improper Riemann integral

$$\int_1^\infty t^{s-1}e^{-ct}\,\mathrm{d}t$$

converges. If the real part of s is greater than 0, prove the improper Riemann integral

$$\int_0^\infty t^{s-1}e^{-ct}\,\mathrm{d}t$$

converges. *When $c = 1$ this integral defines the Γ function.*

Proof. For the first part, it suffices to show that the integrals $\int_1^b t^{s-1}e^{-ct}\,\mathrm{d}t$ is Cauchy; equivalently, it suffices to show that when b_1, b_2 are sufficiently large, $\int_{b_1}^{b_2} t^{s-1}e^{-ct}\,\mathrm{d}t$ can be made arbitrarily small. Indeed, since $\lim_{t\to\infty}|t^s|/e^{ct/2} = \lim_{t\to\infty} t^{\mathfrak{Re}(s)}/e^{ct/2} = 0$ (as given by the hint), we are able to find b_1, b_2 large enough such that $|t^{s-1}| \leqslant e^{ct/2}$. Then we can bound the integral by

$$\int_{b_1}^{b_2} t^{s-1}e^{-ct}\,\mathrm{d}t \leqslant \int_{b_1}^{b_2} |t^{s-1}e^{-ct}|\,\mathrm{d}t \leqslant \int_{b_1}^{b_2} e^{-ct/2}\,\mathrm{d}t.$$

The last term is much nicer, as

$$\int_1^\infty e^{-ct/2}\,\mathrm{d}t = -\frac{2}{c}e^{-ct/2}\bigg|_1^\infty = \frac{2e^{-c/2}}{c} < \infty,$$

which implies $\int_{b_1}^{b_2} e^{-ct/2}\,\mathrm{d}t \to 0$ as $\min(b_1, b_2) \to \infty$. This proves the first claim.

For the second claim, we only need to verify that $\int_0^1 t^{s-1}e^{-ct}\,\mathrm{d}x$ is finite. Since $c > 0$ and $t > 0$, $e^{-ct} < 1$, so

$$\int_0^1 t^{s-1}e^{-ct}\,\mathrm{d}t < \int_0^1 t^{s-1}\,\mathrm{d}t \leqslant \int_0^1 |t^{s-1}|\,\mathrm{d}t = \int_0^1 t^{\mathfrak{Re}(s-1)}\,\mathrm{d}t.$$

By assumption $\mathfrak{Re}(s) > 0$ so $\mathfrak{Re}(s-1) > -1$. Even Calc I students would realize that this integral is finite. □

Problem: 7(b)

For any complex number s with real part > 1, we want to define (a variant of) the Riemann ζ function using the improper Riemann integral formula

$$\xi(s) := \int_0^\infty t^{s/2-1}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right)\mathrm{d}t.$$

Show that this improper Riemann integral exists and is given by

$$\xi(s) = 2\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

where

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,\mathrm{d}t \qquad \zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Proposition

For this problem, you may assume the following statement on nets. Let $(A, \le)$ be a directed set and consider a sequence $f_n(a)$ of a real-valued nets $f_n : A \to \mathbb{R}$. Assume that

(1) for each n, the net $f_n(a)$ converges (in a) to a limit $L_n \in \mathbb{R}$ and

(2) the nets f_n converge uniformly (as a sequence of functions $A \to \mathbb{R}$) to some net $f : A \to R$.

Then L_n converge to some $L \in \mathbb{R}$ and $\lim_a f(a)$ exists and equals L. In other words,

$$\lim_a \lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} \lim_a f_n(a).$$

Proof. Following the hint, we will first consider

$$\int_a^b t^{s/2-1} \left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1 \right) \mathrm{d}t$$

and then let $a \to 0$ and $b \to \infty$. By the first problem, $\sum_{n=-\infty}^{\infty} e^{-\pi t n^2}$ is the limit of a uniformly convergent series of continuous functions and is therefore continuous. Thus the entire integrand is continuous and the integral from a to b is well-defined.

In addition, since $\int$ and $\sum$ for a uniformly convergent series are interchangeable,

$$\int_a^b t^{s/2-1} \left(\sum_{n=-1}^{-\infty} e^{-\pi t n^2} + \sum_{n=1}^{\infty} e^{-\pi t n^2} \right) \mathrm{d}t = 2 \sum_{n=1}^{\infty} \int_a^b t^{s/2-1} e^{-\pi t n^2} \, \mathrm{d}t.$$

Now define A to be the set of pairs of real numbers (a,b) with $0 < a < b$ where $\le$ is defined as $(a,b) \le (a',b')$ if $a' < a < b < b'$. Define

$$f_k(a,b) := 2 \sum_{n=1}^{k} \int_a^b t^{s/2-1} e^{-\pi t k^2} \, \mathrm{d}t.$$

By part (a), since $\mathfrak{Re}(s/2) > 0$, $\lim_{(a,b)\to(0,\infty)} f_k(a,b)$ exists for any k. The net condition (1) is satisfied.

To show our construction of $f_k(a,b)$ also satisfy condition (2), we need to show $\{f_k\}$ is uniformly Cauchy. To do this, we need a u-substitution $u = \pi n^2 t$. For $M > N$ and arbitrary (a,b), this gives

$$\begin{aligned} f_M(a,b) - f_N(a,b) &= 2 \sum_{n=N}^{M} \int_a^b t^{s/2-1} e^{-\pi t n^2} \, \mathrm{d}t \\ &= 2 \sum_{n=N}^{M} \int_{\pi n^2 a}^{\pi n^2 b} \left(\frac{u}{\pi n^2} \right)^{s/2-1} e^{-u} \frac{\mathrm{d}u}{\pi n^2} \\ &= 2\pi^{-s/2} \cdot \int_{\pi n^2 a}^{\pi n^2 b} u^{s/2-1} e^{-u} \, \mathrm{d}u \cdot \sum_{n=N}^{M} \frac{1}{n^s} \\ &< 2\pi^{-s/2} \Gamma(s/2) \sum_{n=N}^{M} \frac{1}{n^s}. \end{aligned}$$

We know that if $\mathfrak{Re}(s) > 1$ then $\sum_{n-1}^{\infty} \frac{1}{n^s}$ converges, so indeed the last term can be made arbitrarily small. This proves that the f_k's are uniformly Cauchy. By the proposition,

$$\lim_{(a,b)\to(0,\infty)} \lim_{k\to\infty} f_k(a,b) = \lim_{k\to\infty} \lim_{(a,b)\to(0,\infty)} f_k(a,b).$$

Evaluating the RHS is highly analogous to the computation above, except this time we actually have $\zeta(s)$ for the last term. Therefore,

$$\xi(s) = 2\sum_{n=1}^{\infty}\int_0^{\infty} t^{s/2-1}e^{-\pi t n^2}\,\mathrm{d}t = 2\pi^{-s/2}\Gamma(s/2)\zeta(s). \qquad \square$$

Problem: 7(c)

Show that the improper Riemann integral

$$\int_1^{\infty} t^{-1}(t^{s/2}+t^{(1-s)/2})\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2}-1\right)\mathrm{d}t$$

makes sense for all complex s.

Proof. We will again adopt the notion of convergent nets here. Just like before, we can rewrite the doubly-infinite series minus 1 as 2 times a singly-infinite series and we can also interchange the summation and integral. Define A to be the set of real numbers > 1 and $b \leq b'$ if $b < b'$, and define

$$f_k(b) := 2\int_1^{b} t^{-1}(t^{s/2}+t^{(1-s)/2})\left(\sum_{n=1}^{k} e^{-\pi t n^2}\right)\mathrm{d}t.$$

Again, by part (a), for any k, $\lim_{b\to\infty} f_k(b)$ exists so the net condition (1) is satisfied.

Now we verify that $f_k(b)$ form a uniformly Cauchy sequence as $k \to \infty$. For $M > N$,

$$\begin{aligned}
f_M(b) - f_N(b) &= 2\sum_{n=N}^{M}\int_1^{b}(t^{s/2-1}+t^{(1-s)/2})e^{-\pi t n^2}\,\mathrm{d}t \\
&= 2\sum_{n=N}^{M} e^{-\pi t n^2/2}\int_1^{b}(t^{s/2-1}+t^{(1-s)/2})e^{-\pi t n^2/2}\,\mathrm{d}t \\
&< 2\sum_{n=N}^{M} e^{-\pi n^2/2}\int_1^{b} At^{K}e^{-\pi t n^2/2}\,\mathrm{d}t \\
&\leqslant 2\sum_{n=N}^{M} e^{-\pi n^2/2}\underbrace{\int_1^{b} At^{(K+1)-1}e^{t\cdot(-\pi n^2/2)}\,\mathrm{d}t}_{\text{finite for each } n}
\end{aligned}$$

where A and K are some constants, as one between $t^{s/2-1}$ and $t^{(1-s)/2}$ will surely decay and the total sum is clearly possible to be bounded by At^K. Since we are looking at a finite sum, we can just leave the integral like that, as long as each one is finite by part (a). It remains to show that $\sum_{n=N}^{M} e^{-\pi m^2/2}$ is finite, which we've shown all the way back in problem 1(b). Therefore the net condition (2), and the proposition gives

$$\int_1^{\infty} t^{-1}(t^{s/2}+t^{(1-s)/2})\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2}-1\right)\mathrm{d}t = 2\sum_{n=1}^{\infty}\int_1^{\infty} t^{-1}(t^{s/2}+t^{(1-s)/2})e^{-\pi t n^2}\,\mathrm{d}t. \qquad \square$$

Problem: 7(d)

Show that for complex s with $\mathfrak{Re}(s) > 1$, we have

$$\xi(s) = -\frac{2}{s(1-s)} + \int_1^\infty t^{-1}(t^{s/2} + t^{(1-s)/2}) \left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1 \right) \mathrm{d}t.$$

Proof. We first split $\int_0^\infty$ into $\int_0^1 + \int_1^\infty$:

$$\xi(s) = \int_0^1 t^{s/2-1} \left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1 \right) \mathrm{d}t + \int_1^\infty t^{s/2-1} \left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1 \right) \mathrm{d}t. \tag{$\mathfrak{M}$}$$

Using u-substitution $u := 1/t$ with $\mathrm{d}u = -t^{-2}\mathrm{d}t$ and the result from problem 1(c), we have

$$\begin{aligned}
\int_0^1 t^{s/2-1} \left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1 \right) \mathrm{d}t &= \int_\infty^1 u^{1-s/2} \left(\sum_{n=-\infty}^{\infty} e^{-\pi n^2/u} - 1 \right) (-t^2)\, \mathrm{d}u \\
&= \int_1^\infty u^{1-s/2} \left(\sum_{n=-\infty}^{\infty} e^{-\pi n^2/u} - 1 \right) u^{-2}\, \mathrm{d}u \\
&= \int_1^\infty u^{-1-s/2} \left(\sum_{n=-\infty}^{\infty} e^{-\pi n^2/u} - 1 \right) \mathrm{d}u \\
[\text{prob } 1(\text{c})] &= \int_1^\infty u^{-1-s/2} \left(\sum_{n=-\infty}^{\infty} \sqrt{u} e^{-\pi n^2 u} - 1 \right) \mathrm{d}u \\
&= \int_1^\infty u^{(-1-s)/2} \left(\sum_{n=-\infty}^{\infty} e^{-\pi u n^2} - \frac{1}{\sqrt{u}} \right) \mathrm{d}u \\
&= \int_1^\infty t^{(-1-s)/2} \left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - \frac{1}{\sqrt{t}} \right) \mathrm{d}t.
\end{aligned} \tag{Υ}$$

Substituting (Υ) into ($\mathfrak{M}$), we see that the difference between their sum and the (big) integral given in the problem is

$$\begin{aligned}
\int_1^\infty t^{(-s-1)/2} \left(1 - \frac{1}{\sqrt{t}} \right) \mathrm{d}t &= \int_1^\infty t^{(-s-1)/2}\, \mathrm{d}t - \int_1^\infty t^{-s/2-1}\, \mathrm{d}t \\
&= \frac{2}{s-1} - \frac{2}{s} = -\frac{2}{s(1-s)}.
\end{aligned}$$

This proves the claim. □

Problem: 7(e)

Show that the zeta function satisfies the function equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s).$$

You may assume that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \qquad \text{for } 0 < \mathfrak{Re}(s) < 1$$

and that

$$\Gamma(s/2)\Gamma((s+1)/2) = \frac{\sqrt{\pi}}{2^{s-1}} \Gamma(s).$$

Proof. Notice from part (d) that $\xi(s) = \xi(1-s)$. Thus,

$$\zeta(s) = \frac{\xi(s)\pi^{s/2}}{2\Gamma(s/2)} \text{ and } \zeta(1-s) = \frac{\xi(1-s)\pi^{(1-s)/2}}{2\Gamma((1-s)/2)} = \frac{\xi(s)\pi^{(1-s)/2}}{2\Gamma((1-s)/2)}.$$

Therefore we can relate $\zeta(s)$ with $\zeta(1-s)$ by

$$\zeta(s) = \frac{\xi(s)\pi^{s/2}}{2\Gamma(s/2)} = \frac{\xi(s)\pi^{(1-s)/2}}{2\Gamma((1-s)/2)} \cdot \frac{\pi^{(2s-1)/2}\Gamma((1-s)/2)}{\Gamma(s/2)}.$$

Since $\sin(\pi s/2) = \dfrac{\pi}{\Gamma(s/2)\Gamma(1-s/2)}$, the cyan term becomes

$$\frac{\pi^{(2s-1)/2}\Gamma((1-s)/2)}{\Gamma(s/2)} = \sin(\pi s/2)\pi^{(2s-3)/2}\Gamma(1-s/2)\Gamma((1-s)/2).$$

Using the second identity given, $\Gamma(1-s/2)\Gamma((1-s)/2) = \dfrac{\sqrt{\pi}}{2^{-s}}\Gamma(1-s)$. Therefore,

$$\begin{aligned}\zeta(s) &= \zeta(1-s) \cdot \sin(\pi s/2) \cdot \pi^{(2s-3)/2} \cdot \frac{\sqrt{\pi}}{2^{-s}} \cdot \Gamma(1-s) \\ &= 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s).\end{aligned}$$

□

Problem: 7(f)

Given the above extension of $\zeta(s)$ to the complex plane, show that $\zeta(s) \neq 0$ unless $s = -2k$ or $\mathfrak{Re}(s) = 1/2$.

Proof. I have discovered a truly remarkable proof which this margin is too small to contain. □